What can Coding do for Control?

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Outline

• Introduction
  – information theory and coding
  – control theory

• Interplay between control and communication
  – coding for interactive communications: tree codes
  – control over noisy channels: anytime capacity

• Construction of linear tree codes
  – existence with high probability
  – efficient decoding for erasure channels
  – examples: stabilizing plants over erasure links, consensus over lossy networks

• Conclusion
  – future work and open problems, *what can coding do for control?*
Single-User Information Theory

Single-user information (Shannon 1948) deals with the study of the fundamental limits of reliable information transmission between a sender and a receiver over a noisy channel.

\[ C = \max_{p_X(\cdot)} \{H(x) + H(y) - H(x, y)\} \]

Key idea: Block coding

- the behavior of the channel over a single use is unpredictable
- but the behavior over many channel uses is:
  - if the channel introduces errors with probability \( p \), say, over \( n \gg 1 \) channel uses it will introduce \( \approx np \) errors
Information Theory Lives in Asymptopia

• start with $b = \{b_i\}_{i=1}^m$ bits (the message), and map them to $c = \{c_i\}_{i=1}^n \in C$ bits (encoding, rate $= \frac{m}{n}$)

• if $y = \{y_i\}_{i=1}^n$ is received, then perform ML decoding:
  $$\hat{c} = \arg \max_{c \in C} p(y|c)$$

If $\frac{m}{n} < C$, there exists a sequence of codes, such that (Shannon)

$$\lim_{n \to \infty} P(\hat{c} \neq c) = 0.$$  

This is nice theory and an elegant result. However,

• it may require unlimited computational resources at the transmitter and receiver (encoding and decoding may need exponential time)

• it assumes asymptotically long delays ($n \to \infty$)
  - encoding can be done only after all the bits $\{b_i\}_{i=1}^n$ are available
  - decoding can be done only after all the outputs $\{y_i\}_{i=1}^n$ are observed
In Practice...

- one cares about the probability-of-error as a function of the rate and the length of the code (error exponents)
- one cares about codes that can be efficiently encoded and decoded
  - algebraic codes
    * Reed-Solomon, Reed-Muller, algebraic geometry
    * Berlekamp-Massey, list-decoding (Guruswamy-Sudan), etc.
  - graph-based codes
    * turbo codes, LDPC codes, expander codes
    * message-passing, bit-flipping, LP decoding, etc.
  - polar codes

In summary, after 60 years of work, we have practical codes that come close to the Shannon limits in many cases.
A Critique...

An early criticism (albeit philosophical) of information was that it did not involve time

- the “information” obtained about knowledge of an event, depends only on the probability of that event

$$\log \frac{1}{p},$$

not on when this knowledge is revealed or when we want to take action on this knowledge

- issue never quite resolved (things like directed mutual information, or the entropy rate of a random process don’t quite cut it)

- problem is that information-theoretic quantities often require some form of ergodicity to have operational significance
• in control theory we observe the output of a dynamical system (plant) and design a controller to regulate its behavior
• controller needs to react and generate control signals \textit{in real-time}
• delay can result in loss of performance and/or instability
• very rich theory has been developed, especially in the LTI case, (LQG control, $H^\infty$ control, Kalman filtering, separation principle)
• increasingly we have applications where systems (autonomous agents, sensor/actuator networks, smart grid, etc.) are remotely controlled and where measurement and control signals are transmitted across noisy channels

• conventional channel coding does not work - the delay is intolerable

• no coding does not work - even optimal control can lead to instability if there is no coding (Sinopoli et al, 2005)
• Take the scalar LTI plant

\[
\begin{align*}
    x_{i+1} &= 2x_i + w_i + u_i \\
    y_i &= x_i + v_i
\end{align*}
\]

where \(w_i\) and \(v_i\) are uniform over \([-1, 1]\).

• Let the channel from the plant to controller be an erasure channel with erasure probability \(\epsilon\).
The Performance of the *Optimal* Controller
Another Example: Distributed Consensus

\[ \begin{align*}
\begin{aligned}
x_{t+1}^{(1)} &= x_t^{(1)} + \alpha (x_t^{(2)} - x_t^{(1)}) \\
x_{t+1}^{(2)} &= x_t^{(2)} + \alpha (x_t^{(1)} - x_t^{(2)}) + \alpha (x_t^{(3)} - x_t^{(2)}) \\
x_{t+1}^{(3)} &= x_t^{(3)} + \alpha (x_t^{(2)} - x_t^{(3)})
\end{aligned}
\end{align*} \]

\[ W = \begin{bmatrix}
1 - \alpha & \alpha & 0 \\
\alpha & 1 - 2\alpha & \alpha \\
0 & \alpha & 1 - \alpha
\end{bmatrix}, \quad \lim_{t \to \infty} W^t = \frac{1}{3} 11^T. \]
Distributed Consensus with Erasures?

\[
\begin{align*}
  x_{t+1}^{(1)} &= x_t^{(1)} \\
  x_{t+1}^{(2)} &= x_t^{(2)} + \alpha (x_t^{(1)} - x_t^{(2)}) + \alpha (x_t^{(3)} - x_t^{(2)}) \\
  x_{t+1}^{(3)} &= x_t^{(3)} + \alpha (x_t^{(2)} - x_t^{(3)}) \\
\end{align*}
\]

\[
W_t = \begin{bmatrix}
  1 & 0 & 0 \\
  \alpha & 1 - 2\alpha & \alpha \\
  0 & \alpha & 1 - \alpha \\
\end{bmatrix}, \quad \prod_t W_t \text{ is not doubly stochastic.}
\]
Network of 20 nodes connected in a line topology, 30% erasures

Without using coding

Agreement reached but not to the initial average

Average of initial Values

number of communication rounds

node values
What to Do?

- the problem is that if we cannot tolerate large delays, we cannot make the noisy channels reliable
- but do we need to do that?
- what do we need to guarantee the stability of the closed loop system?
- what do we need to do in the consensus problem?
Consider a two-party communication system

Alice, $x$

$$s_1 = f_1(x)$$

$$s_2 = f_2(y, s_1)$$

Bob, $y$

$$s_3 = f_3(x, s_1, s_2)$$

Can one do this reliably over noisy links?
Tree Codes (Schulman, 1993)

- semi-infinite \( d \)-ary tree
- each edge labeled by a symbol in an alphabet of size \( d' > d \)
- maps a sequence \( \{s_i\}_{i=1}^{\infty} \) to a sequence \( \{c_i\}_{i=1}^{\infty} \), where \( s_i \in \{0, 1, \ldots, d - 1\} \) and \( c_i \in \{0, 1, \ldots, d' - 1\} \)
- represents a causal code; each path is a codeword; \( R = \frac{\log d}{\log d'} \)
Tree Codes (Schulman, 1993)

- for every pair of paths with a common ancestor and length \( n \), say, we require that the “Hamming distance” between the paths to be at least a fixed proportion of \( n \).
- Schulman proved the existence of tree codes.
- Along with ML decoding, allows reliable interactive communication over a noisy link.
- **Problem:** No explicit constructions; no tractable decoding; existence result is not with high probability.
Tree Codes (Schulman, 1993)

- for the above reasons there has been scant progress in interactive communication over noisy links since Schulman’s work

- **Some very recent results:**
  - *potent tree codes*: relax the requirements of tree codes and hence show existence with high probability; good enough for some problems; not good enough for control (Gelles and Sahai, 2011)
  - improvements to Schulman’s protocol (Braverman and Rao, 2011)
Anytime Capacity (Sahai, 2001)

- scalar unstable LTI system
- noisy channel from plant output to controller
- each measured output quantized to $k$ bits; *causally* encoded and transmitted across channel
- controller attempts to *causally* decode transmitted bits, estimate state of the system, and generate a control signal
- given that we cannot reliably recover the transmitted bits, can we even stabilize the system?
Toy Example: Tracking an Unstable Plant

\[ x_{i+1} = ax_i + w_i, \quad |a| > 1, \quad w_i \in \{-1, 1\} \text{ (unknown)} \]

Assume the initial state \( x_0 = 0 \) is known to the encoder and controller.

- clearly, at each time instant, the encoder should try to convey 1 bit of information to the controller indicating whether \( w_i = 1 \) or \( w_i = -1 \)
- the encoder will causally encode this sequence of bits \( \{b_i\}_{i=0}^{\infty} \) and send them across the channel
- the decoder, at each time instant \( i \), will attempt to decode the entire bit sequence \( \{b_j\}_{j=0}^{i} \) and obtain \( \{\hat{b}_{j|i}\}_{j=0}^{i} \)
Toy Example: Tracking an Unstable Plant

The probability that the first error happens $d$ time steps in the past is:

$$P_e(i, d) = \text{Prob}(\hat{b}_{j|i} = b_j, \forall j < i - d, \hat{b}_{i-d|i} \neq b_{i-d})$$

- Then the mean-square error is bounded by

$$E(x_{i+1} - \hat{x}_{i+1|i})^2 \leq \sum_{d=1}^{i} \left(\frac{a^d - 1}{a - 1}\right)^2 P_e(i, d) < \frac{1}{(a - 1)^2} \sum_{d=1}^{\infty} |a|^{2d} P_e(i, d).$$

- Clearly, if there exists $K, \epsilon$ and $\Delta$, such that for all $i$ and $d > \Delta$:

$$P_e(i, d) < K|a|^{-2d-\epsilon},$$

we will have mean-square stability, i.e., $E(x_{i+1} - \hat{x}_{i+1|i})^2 < \infty$.

- **Remark:** For mean absolute stability, we need $P_e(i, d) < K|a|^{-d-\epsilon}$

**Conclusion:** We do not need arbitrary reliability. Only a reliability that decays appropriately exponentially fast with the delay.
Anytime Capacity

- **Definition:** A channel will be said to have “anytime capacity” $C_{\text{any}}(\lambda)$, for some parameter $\lambda > 1$, if for all rates $R < C_{\text{any}}(\lambda)$, there exists causal encoding and decoding schemes such that

$$P_e(i, d) < K\lambda^{-d-\epsilon}, \quad \forall i, \forall d > \Delta$$

- **Theorem:** Consider a scalar LTI system

$$\begin{aligned}
    x_{i+1} &= \lambda x_i + w_i + u_i \\
    y_i &= x_i + v_i
\end{aligned}$$

where $w_i$ and $v_i$ are bounded disturbances. Then to stabilize this system over a noisy channel it is necessary and sufficient that

1. $k > \log |\lambda|$
2. $R = \frac{k}{n} < C_{\text{any}}(|\lambda|)$

- The theorem is based on the use of tree codes
Some Problems

This is an elegant result, but...

- there are no explicit constructions of tree codes with efficient decoding
- there is very little hope of actually computing $C_{any}(\lambda)$
  - this requires computing optimal error exponents for tree codes
  - even for block codes optimal error exponents have not been computed
- there are no "necessary and sufficient" conditions for systems with vector states

What to do...?
Linear Tree Codes?
Linear Tree Codes

- linear codes can be represented by generator or parity check matrices
- for linear tree codes, these matrices will be (block) lower triangular

\[
\begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3 \\
    \vdots
\end{bmatrix} =
\begin{bmatrix}
    G_{11} & G_{21} & G_{31} \\
    G_{21} & G_{22} & G_{32} \\
    G_{31} & G_{32} & G_{33} \\
    \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    \vdots
\end{bmatrix},
\begin{bmatrix}
    P_{11} & P_{21} & P_{31} \\
    P_{21} & P_{22} & P_{32} \\
    P_{31} & P_{32} & P_{33} \\
    \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3 \\
    \vdots
\end{bmatrix} = 0
\]

where \( b_i \in GF_2^k \), \( c_i \in GF_2^n \), \( G_{ij} \in GF_2^{n \times k} \) and \( P_{ij} \in GF_2^{(n-k) \times n} \)

- Do linear tree codes exist? Requires \( P_e(i, d) < K \lambda^{-d-\epsilon} \), for all \( i \) and \( d > \Delta \). This requires two union bounds (which kills things)—hence Schulman’s approach
Toeplitz Linear Tree Codes

• trick is to make the code Toeplitz

\[
\begin{bmatrix}
G_0 \\
G_1 & G_0 \\
G_2 & G_1 & G_0 \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
P_0 \\
P_1 & P_0 \\
P_2 & P_1 & P_0 \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

• this makes the code “look the same” at all times \(i\), and so we avoid the union bound over \(i\)
Existence of Tree Codes with High Probability

**Theorem:** Choose the entries of the matrices \( \{G_i\}_{i=1}^{\infty} \) (or \( \{P_i\}_{i=1}^{\infty} \)) independently from Bernouli(\( \frac{1}{2} \)) and consider a binary-input channel with Bhattacharya parameter

\[
\zeta = \int_{-\infty}^{\infty} \sqrt{p(y|b=1)p(y|b=0)} dy.
\]

Then with probability \( 1 - 2^{-\Omega(n\Delta)} \), for all rates satisfying

\[
R < 1 - \log(1 + \zeta),
\]

there exists a \( K \) such that the probability of ML decoding satisfies

\[
P_e(i, d) < K2^{-\beta d}, \quad \forall i, \forall d > \Delta
\]

where

\[
\beta < H^{-1}(1 - R) \left( \log \frac{1}{\zeta} + \log(2^{1-R} - 1) \right).
\]
But What to do About ML Decoding?

- to get the error performance we need, we must do ML decoding at each time instant
- for block codes doing this even once is too hard....

However, there is one exception....
But not for Erasure Channels

- Consider a random block linear code with $(N - K) \times N$ parity check matrix $P \left( R = \frac{K}{N} \right)$

- Suppose the codeword $c$ is transmitted and partition it onto the observed entries $c_o$ and the erased entries $c_e$, i.e., $c = \begin{bmatrix} c_o \\ c_e \end{bmatrix}$.

- Now due to the parity check condition

\[
Pc = \begin{bmatrix} P_o & P_e \end{bmatrix} \begin{bmatrix} c_o \\ c_e \end{bmatrix} = 0,
\]

we have

\[
P_e c_e = P_o c_o
\]
•

\[ P_e c_e = P_o c_o \]

If the erasure probability of the channel is \( \epsilon \), then \( c_e \) will have size \( \approx N\epsilon \)

• If \( R = \frac{K}{N} < C = 1 - \epsilon \), then \( N - K > N\epsilon \) and the system of linear equations \( P_e c_e = P_o c_o \), will, with high probability, have a unique solution.

**Conclusion:** For erasure channels ML decoding is simply matrix inversion
**But What About the Lower Triangular Case?**

- for tree codes the matrices $P_e$ and $P_o$ are lower triangular
- therefore even though the matrix $P_e$ has more rows/equations $(N - K)$ than columns/unknowns ($\approx N\epsilon$), the system of equations

\[ P_e c_e = P_o c_o, \]

will most likely *not* have a unique solution (otherwise tree codes would have the same performance of block codes!)

- however, if the tree code corrects all errors above a delay of $d$, this means that if we partition $c_e = \begin{bmatrix} c_{e1} \\ c_{e2} \end{bmatrix}$, where $c_e$ are all the erased entries with delay more than $d$, we must have

\[ P_e c'_e = P_e c''_e \quad \text{implies} \quad c'_{e1} = c''_{e2} \]
partitioning $P_e$, $P_o$ and $c_o$ similarly, we have

$$
\begin{bmatrix}
P_{e11} & c_{e1} \\
P_{e21} & c_{e2}
\end{bmatrix} =
\begin{bmatrix}
P_{o11} & c_{o1} \\
P_{o21} & c_{o2}
\end{bmatrix}
$$

or

$$
\begin{bmatrix}
P_{e11}c_{e1} \\
P_{e21}c_{e1} + P_{e22}c_{e2}
\end{bmatrix} =
\begin{bmatrix}
P_{o11}c_{o1} \\
P_{o21}c_{o1} + P_{o22}c_{o2}
\end{bmatrix}
$$

pre-multiplying the second set of equations by $P_{e22}^\perp$, the orthogonal complement of $P_{e22}$ yields

$$
\begin{bmatrix}
P_{e1} & c_{e1} \\
P_{e22}^\perp P_{e21}
\end{bmatrix} =
\begin{bmatrix}
P_{o11}c_{o1} \\
P_{e22}^\perp P_{e21}c_{o1} + P_{e22}^\perp P_{o22}c_{o2}
\end{bmatrix}
$$

But this latter system of equations must have a unique solution for $c_{e1}$. 
An Efficient Algorithm

1. suppose at time $i$ the bits up to delay $d$ have not yet been decoded (this happens with probability $P_e(i, d) < K \lambda^{-d}$)

2. for these bits, partition $c = \begin{bmatrix} c_e \\ c_o \end{bmatrix}$ and $P = \begin{bmatrix} P_e & P_o \end{bmatrix}$

3. starting with delays $d' = 1, 2, \ldots, d$ check whether the matrix

$$\begin{bmatrix} P_{e1} \\ P_{e22} P_{e21} \end{bmatrix}$$

has full column rank

4. if so, solve for $c_{e1}$ in the system of equations

$$\begin{bmatrix} P_{e1} \\ P_{e22} P_{e21} \end{bmatrix} c_{e1} = \begin{bmatrix} P_{o11} c_{o1} \\ P_{e22} P_{o21} c_{o1} + P_{e22} P_{o22} c_{o2} \end{bmatrix}$$
5. if this does not happen for any \( d' = 1, 2, \ldots d \), go to next time instant

The expected complexity per time instant is constant:

\[
\sum_{d=1}^{\infty} K' d^3 \lambda^{-d}
\]

Furthermore, the probability that the complexity at any given time instant exceeds \( O(d^3) \) decays as \( O(\lambda^{-d}) \).

**Remark:** With feedback, encoding can also be done with constant expected complexity.
A Scalar Example

- Take the scalar LTI system

\[
\begin{align*}
    x_{i+1} & = 2x_i + w_i + u_i \\
    y_i & = x_i + v_i
\end{align*}
\]

where \( w_i \) is uniform over \([-30, 30] \) and \( v_i \) is uniform over \([-1, 1] \).

- Suppose we want to stabilize this over an erasure channel with erasure probability \( \epsilon = 0.3 \) and that we have \( n = 15 \) bits per measurement at our disposal.

- We need an error exponent \( 2^{-\beta} < \frac{1}{2} \). Using the theorem we can see that we need a rate less than \( R < 0.40 \), i.e., we should quantize the measurements to at most \( k = 5 \) bits.
Figure 1: CDF of LQR costs for different realizations of the codes.
The Vector Case

- Consider an observable and controllable LTI system with characteristic polynomial \( a(z) = z^n + a_1 z^{n-1} + \ldots a_n \).

- Let \( \lambda > 1 \) be the smallest positive number such that the matrix

\[
\begin{bmatrix}
\frac{|a_1|}{\lambda} & 1 & 0 & \ldots & 0 \\
\frac{|a_2|}{\lambda} & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{|a_n|}{\lambda} & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

is stable.

- Then we can (mean-square) stabilize the system by appropriately quantizing each output to \( k \) bits and using a tree code, such that

1. \( k > \log \lambda \)

2. \( P_e(i, d) < K |\lambda_{max}|^{-2d-\epsilon} \) for all \( i \) and \( d > \Delta \), where \( |\lambda_{max}| \) is the largest root of \( a(z) \) (in absolute value).
A Vector Example

\[
\begin{align*}
    x_{i+1} &= \begin{bmatrix} 2 & 1 & 0 \\ 0.25 & 0 & 1 \\ -0.5 & 0 & 0 \end{bmatrix} x_i + w_i + u_i \\
y_i &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_i + v_i
\end{align*}
\]

where \(w_i\) and \(v_i\) are truncated \(N(0, 1)\) normals to lie in \([-2.5, 2.5]\).

\(\lambda_{max} = 2\).

- We would like to stabilize the plant over an erasure channel with \(\epsilon = 0.3\).
- We have \(n = 15\) bits per measurement available.
- We need an exponent \(< \frac{1}{2}\): an application of the theorem shows that we need a rate \(R < 0.43\), hence \(k < 7\).
Figure 2: CDF of LQR costs for different realizations of the codes.
Tree Codes for Interactive Protocols: Distributed Consensus

- As mentioned, tree codes can be used to implement any interactive protocol over noisy links.

- Consider the problem of distributed consensus over a given graph:
  - it is well known that if nodes average their and their neighbors’ values, then one reaches consensus to the true average with a rate given by the second largest eigenvalue of a certain doubly-stochastic matrix.
  - if links are erased symmetrically, consensus to the true average still occurs (Jadbabaie and Morse).

- However, symmetric erasures is not a practically reasonable assumption. When erasures are asymmetric, consensus is still reached, but not to the true average.
Network of 20 nodes connected in a line topology, 30% erasures

\[ R_s \geq 0.175 \]
Protocols over Erasure Networks

Consider a graph with maximum degree $\Delta$ with asymmetric erasure links.

**Theorem** (Sukhavasi and Hassibi, 2012) A tree code of rate $r$ and reliability exponent $\beta$, such that $n\beta > 2 \log(1 + \Delta)$, guarantees:

- exponentially small error probability in protocol length
- a simulation rate of $R_s \geq r \rho(r)$, where

$$
\rho(r) = \max_{s > 0} \left\{ s \left| \frac{(1 - s) n \beta}{2} \geq H(s) + \log(1 + \Delta) \right. \right\}
$$
Some Remarks

• We have developed a universal and efficient method for stabilizing plants, and implementing interactive protocols, over erasure channels

• Erasure channels are perhaps the most practically interesting case: most systems quantize their measurements to some number of bits, put them in packets and send them across a lossy network (where packets will either be received or dropped)

• Not clear how to deal with unbounded noise (say, Gaussian). Seems to require perfect feedback. Not clear if this is important in practice.

• Stabilizing a plant is the first step. Optimizing performance is next.
  – this will require studying the trade-offs between control and communication resources
  – should we quantize coarsely, but heavily protect the bits, or quantize finely and moderately protect the bits?
Other Channels

- It would be interesting to develop efficiently-decodable tree codes for other types of channels, especially, the BSC and the AWGNC.
- These appear to be much more challenging, since ML decoding is out of the question.
  - even in the block coding case, the codes that achieve capacity over these channels (such as LDPCs or polar codes) do not do so with an error exponent
  - for example, polar codes approach capacity with a probability-of-error $e^{-\alpha \sqrt{N}}$
- We care about having an error exponent much more than the rate
  - in the block coding case, decoders that achieve an error exponent are those that can correct a fixed fraction of errors
  - Reed-Solomon codes; LDPC and expander codes with bit-flipping and/or LP decoding
What does Coding do for Control?
it replaces a lossy link with a lossless link (of lower rate), but with
random delay.
Conclusion

- Traditional information theory lives in Asymptopia—not appropriate for real-time constraints
- Control has long ignored information theory...
- Controlling an unstable plant over a noisy channel is one place where the two must meet
  - the key object is a “tree code” (essentially a causal code), rather than a block code
  - the key criterion is the interplay between the rate and the decay of the probability of error as a function of the delay (anytime capacity)