Supermodular Network Games

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Abstract

We study supermodular games on graphs as a benchmark model of cooperation in networked systems. In our model, each agent's payoff is a function of the aggregate action of its neighbors and it exhibits strategic complementarity. We study the *largest* Nash equilibrium which, in turn, is the Pareto optimal equilibrium in the presence of positive externalities. We show that the action of a node in the largest NE depends on its *centrality* in the network. In particular, the action of nodes that are in the k-core of the graph is lower bounded by a threshold that is nondecreasing in k. The main insight of the result is that the degree of a node may not be the right indicator of the strength and influence of a node in the equilibrium. We also consider Bayesian supermodular games on networks, in which each node knows only its own degree. In this setting, we show that the largest symmetric Bayesian equilibrium is monotone in the *edge perspective degree distribution*.

I. Introduction

The last decade has witnessed a tremendous effort in analysis, design, and control of large-scale systems using game-theoretic methods. Such methods are naturally well-suited to studying the global effect of local optimizations by individual elements of a system. In many such systems, network structure among the elements of the system plays a significant role in the nature of local interactions: sensor networks and social networks are both examples of "systems" where local interactions are governed by a network structure.

In this paper we study a class of game-theoretic models where a network governs the local interactions among agents. In particular, we focus on network games in which players experience strategic complementarities from their neighbors' actions; formally, the payoff of a node exhibits *increasing differences* in their own action and the action of their neighbors. The games we consider are a special class of *supermodular games*, and as such exhibit a range of useful properties for analysis: pure strategy Nash equilibria are guaranteed to exist; myopic best response dynamics converge; and structural characterizations of equilibria are available.

Many of results are developed in a further specialized class of models where agents' payoffs are *increasing* in the actions of their neighbors; i.e., the payoffs exhibit *positive externalities*. These types of games can be used to capture efforts of nodes to coordinate across the network. For example, such games can be used for models of technology adoption and price competition in networks. In these games, the *largest* Nash equilibrium (in a lattice sense) is also the equilibrium most preferred by all agents.

We develop results for two classes of games, depending on the amount of information nodes have about the network. We first consider a setting where nodes have complete information about the network. We show a close relationship between the largest equilibrium and the set of k-cores of a graph; this relationship is exploited to help shed light on the importance of centrality of a node in the graph in determining the nature of the largest equilibrium. Centrality measures may be poorly estimated by the degree. Hence, predicting the outcome of the game based on the degree profile can be misleading.

Next, we consider a class of games where agents do not necessarily know the network structure. We model this uncertainty using a game of incomplete information, studied via Bayesian equilibria. In the model we consider, each node knows its own degree, but not the degree of other nodes in the graph. The Bayesian equilibria of such games are known to be monotone in the degree; we show that in fact the best response mapping preserves monotonicity of strategies. We use this fact to show that the largest NE is monotone in the edge perspective distribution of the graph. Notably, the first order stochastic dominance ordering for the edge perspective distribution is *not* the same as the first order stochastic dominance ordering for the degree distribution of the graph.

We note that local interactions have also been studied from a probabilistic standpoint. For instance, a simple probabilistic diffusion model can be used to model the spread of viruses in computer networks or contagion in human networks: at each time step, a susceptible host becomes infected with some probability if at least one of its neighbors is infected. The evolution of such non-strategic models of spread is well understood. However, a similar understanding of the role of network structure on equilibrium configurations in a strategic setting are not as well understood; this literature is relatively young. Our paper contributes further to understanding these types of strategic models on networks.

The remainder of the paper is organized as follows. In Section II, we introduce our model and give some examples of network games that can be studied in our framework. Next, we review some of the basic results on the theory of supermodular games in Section III. Section IV is devoted to characterizing the largest NE under the complete information assumption. The incomplete information scenario is analyzed in Section V. We finish the paper with a few qualitative insights and comments about the results in Section VI.

A. Related Work

The abstract notion of games on graphs was introduced in [9] to model a large of number of strategic agents who interact locally. In this work, each player plays a 2 person game with each of its neighbors. The games they consider are not necessarily supermodular, and the emphasis is on computational approaches to find equilibria [13].

By contrast, our work focuses on games on networks that are supermodular. An important example of supermodular games on graphs are technology adoption games. These games have been studied extensively in the social network literature [10], [6], [12]. Roughly speaking, these games model the social phenomenon that people prefer to adopt technologies that are compatible (or equal) to the ones adopted by their acquaintances. The most studied question in this literature is whether a newly introduced technology that is initially adopted by a small number of nodes can eventually be adopted by the majority of people and become epidemic. In [12], Morris gives graph theoretic conditions under which a new technology becomes epidemic. However, little is known for the setting that none of the technologies become epidemic in equilibria.

The most closely related work to our own is [5] and [14]; these papers consider games with local interactions where each player has incomplete information about the network structure. In [14], each user's action is whether to or not to adopt the existing technology. Each player knows its own degree, and has different posterior distributions on the degree of neighboring and non-neighboring nodes. The existence of Bayesian equilibria and monotonicity in the degree are established. This model is an example of supermodular network games in which the action space is finite and has cardinality 2. A more general setting has been studied in [5], where the action space can be either a finite set or a compact interval. Similar results on the existence and monotonicity of BNEs have been presented in this work. Our model in Section V, is closely related to the models considered in [5]. However, our main insight is to show that equilibria are monotone in the edge perspective degree distribution.

II. MODEL

We consider a network represented by $\mathcal{G} = (V, E)$. Each player is associated with a node in $\mathcal{G}$ and is limited to play only with its neighbors. All players have the same strategy set $\mathcal{X} = [0, 1]$. (We can generalize our model and results to include discrete, finite action sets as well in a straightforward manner; in this paper we only present the results for continuous action spaces.) Let $V = \{1, 2, \ldots, n\}$ and let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denote the action vector. The payoff of player i given $\boldsymbol{x}$ is:

$$\Pi_i(\boldsymbol{x}) = u(x_i, \sum_{j \in \partial i} x_j) - c(x_i) \tag{1}$$

where $\partial i = \{j \in V : (i, j) \in E\}$. In words, each player's *utility* depends on its own action and the aggregate action of its neighbors. However, the player's *cost* is only a function of its own action. In the sequel, the following notation will be used to represent the sum of actions of i's neighbors:

$$\Sigma_i(\boldsymbol{x}_{-i}) = \sum_{j \in \partial i} x_j.$$

We assume that $\Pi_i(\boldsymbol{x})$ is continuous, and has *increasing differences* in x_i and $\boldsymbol{x}_{-i}$: for all $x_i > x_i'$, and $\boldsymbol{x} \geq \boldsymbol{x'}$,

$$\Pi_i(x_i, \boldsymbol{x'}_{-i}) - \Pi_i(x_i', \boldsymbol{x'}_{-i}) \geq \Pi_i(x_i, \boldsymbol{x}_{-i}) - \Pi_i(x_i', \boldsymbol{x}_{-i}). \tag{2}$$

If $u(x_i, \Sigma_i(\boldsymbol{x}_{-i}))$ is increasing in the second argument, we say that the payoff function exhibits *positive externalities*. We do not require the positive externalities assumption for all our results; however, as explained in section III, imposing this assumption leads to stronger conclusions about the game.

We conclude this section with some examples.

Example 2.1: **(Bertrand Competition)** Suppose each node represents a firm. Each firm competes with its neighbors and sets a price $p_i \in [0,1]$. Assume that firm i receives demand $D_i(p_i, \boldsymbol{p}_{-i})$. If all firms share the same marginal cost c per unit of good produced, then:

$$\Pi_i(\boldsymbol{p}) = (p_i - c) D_i(p_i, \boldsymbol{p}_{-i}).$$

We assume $D_i(p_i, \boldsymbol{p}_{-i}) = a - bp_i + d \sum_{j \in \partial i} p_j$ is the demand that firm i receives. Since each node only competes with its neighbors, the network structure plays an important role in the price of a firm at equilibrium. It is easy to check that these utilities exhibit increasing differences in p_i and $\boldsymbol{p}_{-i}$. This game is studied in detail in Section IV-A.

Example 2.2: **(Local Search Model)** Suppose each node represents an agent who is looking for trading partners. Each agent i exerts effort $e_i \in [0,1]$ that costs $c(e_i)$, where $c(\cdot)$ is increasing and continuous. Node i's effort only affects the payoff of its neighbors. In other words, the effort of a non-neighbor agent does not help in finding trading partners. The probability of finding a partner is $e_i \sum_{j \in \partial i} e_j$. Hence,

$$\Pi_i(\boldsymbol{e}) = e_i \sum_{j \in \partial i} e_j - c(e_i). \tag{3}$$

This is a simplified version of Diamond's search model [3] modified to incorporate the network structure.

III. Definitions and Preliminaries

In this section, we first define *supermodular games* and then review some of the basic properties of this class of games. The theory of supermodular games is closely related to the theory of lattices; in general, supermodular games are defined for any strategy set that is a complete lattice in k-dimensional Euclidean space. However, since we assume that the $\mathcal{X} = [0,1]$, we present a simplified version of the results. We refer the reader to [15] for a thorough review of the theory.

Definition 3.1: Consider a finite n-player game, where player i has action set $\mathcal{X}$ and payoff function Π_i, and $\mathcal{X}$ is a compact subset of $\mathbb{R}$. The resulting game is *supermodular*, if:

1) Fixing $\boldsymbol{x}_{-i}$, $\Pi_i(x_i, \boldsymbol{x}_{-i})$ is upper semicontinuous in x_i, and fixing x_i, $\Pi_i(x_i, \boldsymbol{x}_{-i})$ is continuous in $\boldsymbol{x}_{-i}$ with a finite uniform bound in $\boldsymbol{x}_{-i}$.
2) $\Pi_i(x_i, \boldsymbol{x}_{-i})$ has increasing differences in x_i and $\boldsymbol{x}_{-i}$.

Clearly, the general class of games defined in (1) satisfies the above conditions, and therefore such games are supermodular. We can also extend the above definition to include other parameters that affect the payoff. Given a partially ordered set T, the tuple $(n, \mathcal{X}^n, \Pi_i(\boldsymbol{x}, t))$ is a *family of supermodular games indexed by* t if $\Pi_i(\boldsymbol{x}, t)$ has increasing differences in $\boldsymbol{x}$and t.

The following proposition is a key property of supermodular games. See [15] for details.

Proposition 3.2: Given $\boldsymbol{x}$, for each player i, define the largest best response mapping as follows.

$$\overline{\mathrm{BR}}_i(\boldsymbol{x}_{-i}) = \max \left\{ \underset{z \in [0,1]}{\operatorname{argmax}} \left[u(z, \boldsymbol{x}_{-i}) - c(z) \right] \right\};$$

If $\boldsymbol{x}_{-i} \geq \boldsymbol{x'}_{-i}$ then $\overline{\mathrm{BR}}_i(\boldsymbol{x}_{-i}) \geq \overline{\mathrm{BR}}_i(\boldsymbol{x'}_{-i})$.

The following corollaries are consequences of Proposition 3.2 [15].

Corollary 3.3: Pure strategy Nash equilibria exist in supermodular games.

Moreover, the largest NE, $\overline{\boldsymbol{x}}^*$ ($\underline{\boldsymbol{x}}^*$) can be computed by following the largest best response dynamics started from an appropriate point, as follows. Suppose $\boldsymbol{x}^0 \geq \overline{\boldsymbol{x}}^*$; we define the largest best response dynamics as follows,

$$\boldsymbol{x}^{t+1} = (\overline{\mathrm{BR}}_1(\boldsymbol{x}^t_{-1}), \overline{\mathrm{BR}}_2(\boldsymbol{x}^t_{-2}), \ldots, \overline{\mathrm{BR}}_n(\boldsymbol{x}^t_{-n}))$$

and we have:

$$\lim_{t \to \infty} \boldsymbol{x}^t = \overline{\boldsymbol{x}}^*$$

In particular, in our setting if $\boldsymbol{x}^0 = (1, 1, \ldots, 1)$ then $\boldsymbol{x}^t \downarrow \overline{\boldsymbol{x}}^*$.

Corollary 3.4: For a family of supermodular games indexed by t, $\overline{\boldsymbol{x}}^*(t)$ is monotone in t, i.e., $\overline{\boldsymbol{x}}^*(t) \geq \overline{\boldsymbol{x}}^*(t')$ for $t \geq t'$.

Definition 3.5: Suppose $\boldsymbol{x}, \boldsymbol{x'} \in \mathcal{X}^n$. We say $\boldsymbol{x}$ is *Pareto preferred* between $\boldsymbol{x}$ and $\boldsymbol{x'}$ if for each i, $\Pi_i(\boldsymbol{x}) \geq \Pi_i(\boldsymbol{x'})$.

The subsequent sections are devoted to studying the largest NE. The following proposition illustrates an important property of the largest NE when each player feels positive externality from other players.

Proposition 3.6: For a supermodular game, if each payoff function exhibits positive externalities then the pure Nash equilibria are ordered in accordance with Pareto preference. In particular, the largest NE, $\overline{\boldsymbol{x}}^*$ is Pareto preferred among the set of all Nash equilibria.

Proof: Suppose, $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are two equilibria of the game and $\boldsymbol{x}^* \leq \boldsymbol{y}^*$. The following inequalities establish the thesis,

$$\boldsymbol{x}^* \leq \boldsymbol{y}^* \Rightarrow \boldsymbol{x}^*_{-i} \leq \boldsymbol{y}^*_{-i} \Rightarrow \Pi_i(x_i, \boldsymbol{x}^*_{-i}) \leq \Pi_i(x_i, \boldsymbol{y}^*_{-i}) \leq \Pi_i(y_i, \boldsymbol{y}^*_{-i}) \tag{4}$$

The last inequality follows from the fact that $\boldsymbol{y}^*$ is a Nash equilibrium. ■

In Section V, we also consider Bayesian supermodular games; in such models, each agent i effectively plays against a random vector $\boldsymbol{X}_{-i}$. We require the following definition.

Definition 3.7: Suppose $X,\ X' : \Omega \to \mathbb{R}$ are two real valued random variables. X' is *first order stochastically dominated* by X if for $x \in \mathbb{R}$,

$$\mathbb{P}(X' \leq x) \geq \mathbb{P}(X \leq x) \tag{5}$$

we use the notation $X \succeq X'$ to denote the first order stochastic dominance (FOSD). An equivalent condition to 5 is that for every nondecreasing function $f(\cdot)$, $\mathbb{E}[f(X)] \geq \mathbb{E}[f(X')]$.

Proposition 3.2 generalizes to the abstract setting that each player i plays against a random vector $\boldsymbol{X}_{-i}$, under the FOSD ordering on distributions. First we define the corresponding largest *best expected response* as follows:

$$\overline{\mathrm{BER}}_i(\boldsymbol{X}_{-i}) = \max \left\{ \underset{z \in [0,1]}{\operatorname{argmax}}\, \mathbb{E}\left[u(z, \boldsymbol{x}_{-i}) - c(z)\right] \right\} \tag{6}$$

Proposition 3.8: If $\boldsymbol{X}_{-i} \succeq \boldsymbol{X'}_{-i}$ then, $\overline{\mathrm{BER}}_i(\boldsymbol{X}_{-i}) \geq \overline{\mathrm{BER}}_i(\boldsymbol{X'}_{-i})$.

IV. Complete Network Information

In this section we study supermodular games where nodes have complete information about the network. We require the following definition.

Definition 4.1: Given $\mathcal{G}$, the induced subgraph $\mathcal{G}_k$ is the k-core of $\mathcal{G}$ if it is the largest subgraph such that the degree of all nodes in $\mathcal{G}_k$ is at least k.

Cores can be used as a measure of centrality in the network. Note that k-cores can be obtained by a simple pruning algorithm: at each step, we remove all nodes with degree less than k. We repeat this procedure until there exist no such nodes or all nodes are removed. It is not hard to see that this algorithm converges in at most n steps. Hence, we can determine all possible cores in at most $\Delta_{max} n$ iterations, where Δ_{max} is the maximum degree of nodes in V.

We define the *coreness* of each node as follows.

Definition 4.2: The *coreness* of node i, cor(i), is k if and only if $i \in \mathcal{G}_k$ and $i \notin \mathcal{G}_{k+1}$.

It is not hard to see that cor$(i) \leq$ deg(i). However, there is no other relation between the degree and coreness of nodes in a graph. Figure 1(a) is an example of a graph in which deg$(1) >$ deg(2) but cor$(1) <$ cor(2).

We have the following theorem, which relates the largest NE to the k-cores of the graph.

Theorem 4.3: Given a graph $\mathcal{G}$, let $K = \max_k\{\mathcal{G}_k \neq \emptyset\}$. For the game defined by (1), the largest NE, $\overline{\boldsymbol{x}}^*$, has the following properties:

1) There exist thresholds $0 \leq \gamma_1 \leq \gamma_2 \cdots \leq \gamma_K$ such that if cor$(i) = k$, then $\overline{x}_i^* \geq \gamma_k$.
2) For a graph $\mathcal{G}' = (V, E')$ such that $E' \subseteq E$, let $\overline{\boldsymbol{y}}^*$ be the largest NE. then $\overline{\boldsymbol{y}}^* \leq \overline{\boldsymbol{x}}^*$.

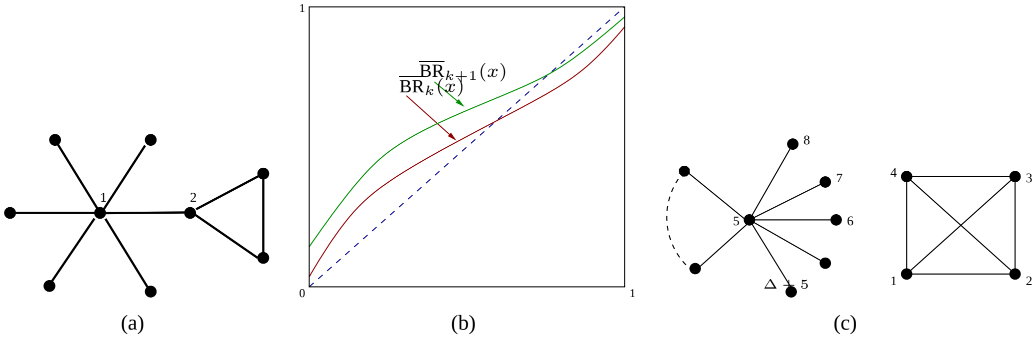

Fig. 1. (a) $\mathrm{cor}(1) = 1$; $\mathrm{cor}(2) = 2$. (b) Monotonicity of fixed points of $\overline{BR}_k(x)$. (c) Graph of Example 4.4.

Proof: **Property (1)**: Let $\overline{\mathrm{BR}}_k : [0,1] \to [0,1]$ be the largest best response to kx, i.e.,

$$\overline{\mathrm{BR}}_k(x) = \max\left\{ \underset{z\in[0,1]}{\mathrm{argmax}} \left[u(z,kx) - c(z)\right]\right\}$$

It is straightforward to show that $\overline{\mathrm{BR}}_k$ is a nondecreasing mapping from $[0,1]$ to itself, and so by Tarski's fixed point theorem, there exists $\gamma \in [0,1]$ such that $\overline{\mathrm{BR}}_k(\gamma) = \gamma$ [15]. According to Proposition 3.3, if we let $\boldsymbol{x}^0 = (1,1,\ldots,1)$ and follow the largest best response mapping, we will converge to the largest NE. Consider $i \in \mathcal{G}_k$, and let x_i^t be the t-step iterate in this sequence. After one step,

$$x_i^1 = \overline{\mathrm{BR}}_{\mathrm{deg}(i)}(1) \geq \overline{\mathrm{BR}}_k(1). \tag{7}$$

Further, node i has at least k neighbors for which an inequality similar to (7) holds. Therefore,

$$\Sigma_i(\boldsymbol{x}^1) \geq k\overline{\mathrm{BR}}_k(1).$$

Applying Proposition 3.2 results in:

$$x_i^2 \geq \overline{\mathrm{BR}}_k(\overline{\mathrm{BR}}_k(1)) = \overline{\mathrm{BR}}_k^{(2)}(1).$$

Repeating the above argument, we have:

$$x_i^t \geq \overline{\mathrm{BR}}_k^{(t)}(1),\ t \geq 0.$$

Therefore,

$$\overline{x}_i^* = \lim_{t\to\infty} x_i^t \geq \lim_{t\to\infty} \overline{\mathrm{BR}}_k^{(t)}(1).$$

Proposition 3.2 implies that $\overline{\mathrm{BR}}_k$ is a nondecreasing function. Therefore, $\overline{\mathrm{BR}}_k^{(1)}(1) \leq 1$ implies $\overline{\mathrm{BR}}_k^{(t+1)}(1) \leq \overline{\mathrm{BR}}_k^{(t)}(1)$. Hence $\lim_{t\to\infty} \overline{\mathrm{BR}}_k^{(t)}(1) = \gamma_k$, where γ_k is the largest fixed point of $\overline{\mathrm{BR}}_k(x)$. The only remaining task is to show that $\gamma_k \leq \gamma_{k+1}$. This follows from the observation that $\overline{\mathrm{BR}}_k(x) \leq \overline{\mathrm{BR}}_{k+1}(x)$ and therefore its corresponding fixed point, γ_{k+1}, is at least γ_k, Figure 1(b).

Property (2): Let $\mathcal{E}_{\mathcal{G}}$ be the collection of all subsets of $E_{\mathcal{G}} = \{(i,j),\ i,j \in V\}$. It is straightforward to check that $\mathcal{E}_{\mathcal{G}}$ is partially ordered under set inclusion. Let

$$\Sigma_i'(\boldsymbol{x}) = \sum_{j\in\partial' i} x_j,$$

where $\partial' i = \{j \in V, (i,j) \in E'\}$. Since $\boldsymbol{x} \geq 0$, $\Sigma_i'(\boldsymbol{x}) \leq \Sigma_i(\boldsymbol{x})$. Equation (2) implies that $\Pi_i(\boldsymbol{x})$ satisfies the increasing differences property with respect to the partially ordered set $\mathcal{E}_{\mathcal{G}}$. Considering the set of edges as the parameter, the thesis follows by applying Corollary 3.4. ■

Note that monotonicity of the thresholds $\gamma_1, \gamma_2, \ldots, \gamma_K$ does not imply monotonicity of the actions in the largest NE. In particular, depending on the network structure, we may have games in which $\overline{x}_i^* > \overline{x}_j^*$ while cor$(i) <$ cor(j). An instance of such a game is illustrated in the following example.

Example 4.4: Let $V = \{1, 2, \ldots, \Delta + 5\}$, $E = \{(i,j),\ 1 \le i < j \le 4\} \cup \{(5,6), (5,7), \ldots, (5,\Delta)\}$, and

$$\Pi_i(\boldsymbol{x}) = x_i \Sigma_i(\boldsymbol{x}) - x_i^3 - \frac{3}{4} x_i.$$

The graph is shown in Figure 1(c). The intuition is that even though cor$(5) = 1$, it has so many weekly connected neighbors that in equilibrium, $\sum_{j \in \partial 5} \overline{x}_j^* > \sum_{j \in \partial 1} \overline{x}_j^*$. In particular, simple algebra shows that if $\frac{1}{\sqrt{3}} \sqrt{\frac{\Delta}{2\sqrt{3}} - 1} > 1$ then,

$$\overline{x}_i^* = \begin{cases} \frac{2-\sqrt{3}}{2} & 1 \le i \le 4; \\ 1 & i = 5; \\ \frac{1}{2\sqrt{3}} & i \ge 6. \end{cases}$$

Hence, cor$(1) >$ cor(5); however, $\overline{x}_1^* < \overline{x}_5^*$.

The following corollary establishes an upper bound on the distance between the largest action vector, i.e., $\mathbf{1}$ and the largest NE using the k-core sizes and the thresholds derived in Theorem 4.3. Suppose that $\mathbf{1}$ is a Nash equilibrium if the graph is complete, i.e., $u(1, n-1) - c(1) \ge u(x, n-1) - c(x),\ x \in [0,1]$. The following corollary measures the distance between the largest NE of the game on the complete graph and the one on graph $\mathcal{G}$.

Corollary 4.5: Let $\|\boldsymbol{x}\|_1$ be the L_1 norm of $\boldsymbol{x}$ and $|\mathcal{G}_k|$ be the number of nodes in $\mathcal{G}_k$. Then:

$$\frac{\|\mathbf{1} - \overline{\boldsymbol{x}}^*\|_1}{n} \le \sum_{k=1}^{K} (1 - t_k) \frac{|\mathcal{G}_k \setminus \{\bigcup_{k'=k+1}^{K} \mathcal{G}_{k'}\}|}{n}. \tag{8}$$

Inequality (8) is a direct consequence of Theorem 4.3 and the fact that the cores of a graph are nested, i.e., $\mathcal{G}_{k+1} \subseteq \mathcal{G}_k$.

A. Coreness and Bonacich Centrality

In this section, we consider a specific quadratic payoff function and explicitly calculate the Nash equilibria of the game. In this setting, the largest NE, which is indeed the unique pure NE, is related to Bonacich centrality of the network. Bonacich centrality is another measure of centrality that weighs the neighbors by their centrality in the network [2].

Definition 4.6: Let $A_{\mathcal{G}}$ denote the adjacency matrix of graph $\mathcal{G}$. For each node i, its *Bonacich centrality of parameter* α, $\mathcal{B}_i(\alpha)$, is $\left((I - \alpha A_{\mathcal{G}})^{-1}\mathbf{1}\right)_i$. The matrix inversion is well defined for $\alpha < \frac{1}{|\lambda_{\max}|}$, where $\lambda_{\max}$ is the maximum eigenvalue of $A_{\mathcal{G}}$.

We consider a specific model where:

$$\Pi_i(\boldsymbol{x}) = x_i \sum_{j \in \partial i} x_j - \frac{b}{2} x_i^2 + a x_i,$$

where $a \ge 0$, and $0 < b < |\lambda_{\max}|$. This game is supermodular and has a unique pure Nash equilibrium:

$$\boldsymbol{x}^* = \frac{a}{b} \mathcal{B}\left(\frac{1}{b}\right) \tag{9}$$

The above calculation is straightforward and similar to the calculation in [17] and [7]. Note that if we assume that $\boldsymbol{x} \in [0, K]^n$ where $K > \max_i x_i^*$, the result will not change and then by scaling the strategies this game satisfies all the assumptions made in Section II.

Bonacich centrality can be interpreted as follows:

$$\mathcal{B}(\alpha) = (I - \alpha A_{\mathcal{G}})^{-1} \mathbf{1} = \left(\sum_{l=0}^{\infty} \alpha^l A_{\mathcal{G}}^l\right) \mathbf{1}. \tag{10}$$

Here $A^l_{\mathcal{G}}(ij)$ is the total number of walks of length l between nodes i and j. Consequently, $\left(A^l_{\mathcal{G}}\mathbf{1}\right)_i$ is the total number of walks of length l between i and any nodes in the network. The term $\left(A^l_{\mathcal{G}}\mathbf{1}\right)_i$ can be written as,

$$\left(A^l_{\mathcal{G}}\mathbf{1}\right)_i = \sum_{j\in\partial i}\left(A^{l-1}_{\mathcal{G}}\mathbf{1}\right)_j. \tag{11}$$

In words, each step is a *weighted* sum of the neighbors. The weights are computed recursively and $\left(A_{\mathcal{G}}\mathbf{1}\right)_i = \text{deg}(i)$. The following lemma establishes the monotonicity of Bonacich centrality of a node with respect to its coreness.

Lemma 4.7: If $\text{cor}(i) = k$ then $\mathcal{B}(\alpha)_i \geq \frac{1}{1-\alpha k}$.

Proof: First note that $A_{\mathcal{G}}$ is a nonnegative matrix; hence by the Perron-Frobenius theorem, $\lambda_{\max}$ is real and positive, and at most $\max_i \sum_j A_{\mathcal{G}}(ij)$. Observe that $\text{cor}(i) = k$ implies that $k \leq \max_i \sum_j A_{\mathcal{G}}(ij)$ and $\alpha k < 1$, hence the matrix inversion is well defined. Similar to (11),

$$\mathcal{B}_i(\alpha) = 1 + \alpha\text{deg}(i) + \alpha^2\sum_{j\in\partial i}\text{deg}(j) + \alpha^3\sum_{j\in\partial i}\sum_{h\in\partial j}\text{deg}(h) + \ldots$$

Further, $\text{cor}(i) = k$ implies that $\text{deg}(i) \geq k$ and i has at least k neighbors who have at least k neighbors. Therefore, $\sum_{j\in\partial i}\text{deg}(j) \geq k^2$. Repeating this argument results in:

$$\mathcal{B}_i(\alpha) \geq \sum_{l=0}^{\infty}\alpha^l k^l = \frac{1}{1-\alpha k}.$$

■

We conclude by noting that the inequality in the preceding result is tight when i belongs to an isolated clique of size k. Roughly speaking higher coreness implies higher Bonacich centrality. Therefore, we expect that nodes with a higher Bonacich centrality take higher actions at the equilibrium. It is easy to see that Bonacich centrality is not directly related to the degree as well.

V. Incomplete Network Information

In a large network, it is unrealistic to assume that each node knows the whole structure of the graph. Moreover, the largest best response dynamic may not converge fast enough. These considerations motivate us to define a game with *incomplete information* and study its Bayesian equilibria. Nature generates the network according to a given degree distribution which is public information. Each player i observes its own set of neighbors ∂i. However, it does not know the set of neighbors of any other node in the graph. Hence, it plays against a random strategy vector.

The correct mathematical model for such a setting is one where nature chooses a random network with a given degree distribution. For this purpose we consider the *configuration model*, constructed as follows [4]. Given n nodes, let d_i the degree of node i, chosen according to a probability distribution with $\mathbb{P}(d_i = k) = p_k$; the distribution $(p_0, \ldots, p_R)$ is called the *degree distribution*. (We assume the degrees are bounded above by R.) We assign d_i half-edges to node i, and then connect together half-edges between nodes uniformly at random. If we condition on the sum of degrees of nodes being even, and no multiple edges or self-loops, the resulting probability distribution is uniform over all graphs on n nodes drawn from the given degree distribution [4].

What is the degree of a neighbor of node i? In general, the degrees of i's neighbors may be correlated with i's degree; however, this correlation vanishes as the number of nodes becomes large. For the remainder of the section we make the approximation that *the degree of a neighbor of node i is independent of node i's degree*. (This approximation can be made rigorous using calculations similar to Chapter 9 of [11].) Note that nodes are more likely to connect to nodes of higher degree; in fact, under the configuration model,

$$\mathbb{P}(\text{an edge is incident to a node of degree } r) \approx \frac{rp_r}{\sum_{r=0}^{R} rp_r} = p^e_r.$$

We refer to the distribution $(p^e_0, \ldots, p^e_R)$ as the *edge perspective degree distribution*. The approximation in the preceding expression has an $o(n)$ error, due to the fact that the degree of the neighbor is not independent of a given node's degree.

As noted above we consider a simplified model where nodes assume their neighbors' degrees are i.i.d. and drawn from the edge perspective distribution. In our setting, the resulting network game is anonymous, meaning that the identity of players does not have any effect on the payoffs. Therefore, node i only needs to know its own degree. A strategy in this game denotes node i's action as a function of its degree. The strategy vector $(s^*(0), s^*(1), \ldots, s^*(R))$ is a *symmetric Bayesian equilibrium* if:

$$\mathbb{E}[\Pi_i(x_i = s^*(r), \boldsymbol{S}^*_{-i})|\text{deg}(i) = r] \geq \mathbb{E}[\Pi_i(x_i = x, \boldsymbol{S}^*_{-i})\text{deg}|(i) = r], \quad x \in [0,1], 0 \leq r \leq R. \tag{12}$$

The above setting has been studied in [5] without explicitly characterizing the *belief* of node i about the degree of its neighbors. The following proposition is a special case of Proposition 2 of [5].

Proposition 5.1: Let $\overline{\boldsymbol{s}}^*$ denote the largest BNE of the game described above. If $r \geq r'$ then $\overline{s}^*(r) \geq \overline{s}^*(r')$. We present an alternative proof that exploits the properties of the dynamics that converge to the largest BNE.

Proof: Suppose $\text{deg}(i) = r_i$ and all the nodes are playing according to strategy vector $\boldsymbol{s}$. The expected payoff of node i by playing action x_i is:

$$\begin{aligned}
&\mathbb{E}[\Pi_i(x_i, \boldsymbol{S}_{-i})|\text{deg}(i) = r_i] \\
&= \sum_{(r_1,\ldots,r_{r_i}) \in \{0,\ldots,R\}^{r_i}} \Pi\left(x_i, (s(r_1), \ldots, s(r_{r_i}))\right) \mathbb{P}\left((\text{deg}(j_1), \ldots, \text{deg}(j_{r_i})) = (r_1, \ldots, r_{r_i})|\text{deg}(i) = r_i\right)
\end{aligned}$$

Incorporating the independent degree assumption, we have:

$$\mathbb{E}[\Pi_i(x_i, \boldsymbol{S}_{-i})|\text{deg}(i) = r] = \sum_{(r_1,\ldots,r_{r_i}) \in \{0,\ldots,R\}^{r_i}} \Pi_i(x_i, \sum_{r'=1}^{r_i} s(r_{r'})) \prod_{r'=1}^{r_i} p^e(r_{r'})$$

Note that $\mathbb{E}[\Pi_i(x_i, \boldsymbol{S}_{-i})|\text{deg}(i) = r]$ has the form of $\mathbb{E}[\Pi_i(x_i, \sum_{r'=1}^{r_i} S_{r'})]$ where $S_1, S_2, \ldots, S_{r_i}$ are i.i.d. nonnegative random variables. Basic probability shows that if $Y_1, Y_2, \ldots Y_r, Y'_1, Y'_2, \ldots, Y'_{r'}$ are nonnegative i.i.d. random variables and $r \geq r'$, then $\sum_{p=1}^{r} Y_p \succeq \sum_{p=1}^{r'} Y'_p$. Therefore if $r_i \geq r'_i$ then $\sum_{r'=1}^{r_i} S_{r'} \succeq \sum_{r'=1}^{r'_i} S_{r'}$. Applying Proposition 3.8 we conclude that $\overline{BER}(\boldsymbol{S}_{-i})$ is monotone in the degree.

To establish the result, we follow the largest best expected response mapping for strategies: Let $\boldsymbol{s}^0 = \mathbf{1}$. Note that $s^1(r) = \overline{BER}(r)$, therefore $s^1(r)$ is monotone in degree. At iteration 2, $s^2(r) = \overline{BER}(S_1 + S_2 + \cdots + S_{r_i})$ where $S_1, S_2, \ldots, S^1_{r_i}$ are i.i.d nonnegative random variables. As explained in the last paragraph $\overline{BER}(\boldsymbol{S}_{-i})$ is monotone in the degree and hence, $s^2(r) \geq s^2(r')$ for $r \geq r'$. By the same argument, $s^t(r) \geq s^t(r')$ for $r \geq r'$. The result follows from the fact that the described dynamics converge to the largest BNE [16]. ■

The next proposition, establishes the monotonicity of the largest BNE in the edge perspective degree distribution.

Proposition 5.2: Consider two edge perspective degree distributions $\boldsymbol{p}^e_1$ and $\boldsymbol{p}^e_2$. Let $\overline{\boldsymbol{s}}^*_1$ and $\overline{\boldsymbol{s}}^*_2$ denote the largest BNE for the games defined by the corresponding degree distributions, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. If $\boldsymbol{p}^e_1 \succeq \boldsymbol{p}^e_2$ then $\overline{\boldsymbol{s}}^*_1 \geq \overline{\boldsymbol{s}}^*_2$.[1]

Proof: Similar to the proof of Proposition 5.1, we follow the largest best expected response mapping for strategies. Suppose $\boldsymbol{s}^t_\alpha$ represents the strategy vectors in step t according to distribution $\boldsymbol{p}^e_\alpha$, with $\alpha = 1, 2$. Let $\boldsymbol{s}^0_\alpha = \mathbf{1}$. Clearly $s^1_1(r) = s^1_2(r) = \overline{BER}(r)$ is monotone in the degree. Basic probability shows that if $Y \succeq Y'$ and $f(\cdot)$ is a nondecreasing function,then $f(Y) \succeq f(Y')$ (this is basically a direct consequence of the equivalent condition in Definition 5.). In our setting, $s^1(\cdot)$ is a nondecreasing function of r. Therefore, $\boldsymbol{p}^e_1 \succeq \boldsymbol{p}^e_2$ implies that $S^1_1 \succeq S^1_2$. Applying Proposition 3.8, we conclude that $\boldsymbol{s}^2_1 \geq \boldsymbol{s}^2_2$. A similar argument can be used to inductively show $\boldsymbol{s}^t_1 \geq \boldsymbol{s}^t_2$ for $t \geq 2$. The result follows from the fact that the described dynamics converge to the largest BNE [16]. ■

Remark Note that in the configuration model, $\boldsymbol{p}^e_1 \succeq \boldsymbol{p}^e_2$ neither implies nor is implied by $\boldsymbol{p}_1 \succeq \boldsymbol{p}_2$. For example $(p_3 = .5, p_5 = .5) \succeq (p_2 = .5, p_4 = .5)$ but $(p^e_3 = 3/8, p^e_5 = 5/8) \not\succeq (p^e_2 = 2/6, p^e_4 = 4/6)$; further, $(p^e_3 = 1/3, p^e_5 = 2/3) \succeq (p^e_2 = 1/2, p^e_4 = 1/2)$ but $(p_3 = 5/11, p_5 = 6/11) \not\succeq (p_2 = 1/3, p_4 = 2/3)$.

Observe that these results are obtained using an asymptotic approximation based on the configuration model. We conjecture that these results can be made rigorous even for large finite graphs; verifying these claims is a topic of ongoing research.

[1] Here $\boldsymbol{p}^e_1 \succeq \boldsymbol{p}^e_2$ means that if X_1 and X_2 have distributions $\boldsymbol{p}^e_1$ and $\boldsymbol{p}^e_2$ respectively then $X_1 \succeq X_2$.

VI. Conclusion

In this paper, we study the effects of network structure on the configuration of the equilibria in network games. Motivated by the growing interest in cooperative networked systems, we consider a game in which each player locally interacts with its neighbors; all the players have a symmetric payoff that exhibits strategic complementarity. Under the positive externality assumption, the largest NE of this game is the Pareto optimal equilibrium. Following the largest best response dynamics, we derive lower bounds on the action of nodes in the equilibrium of interest. In particular, we show that a lower bound on node i's action depends on the coreness of i. The results demonstrate the effect of centrality on the equilibrium structure. Moreover, it implies that even though the game is played locally, the global structure of the network has a significant effect on the equilibrium. In other words, the behaviour of two far away nodes may not be independent. For example, decreasing the degree of a node that is in a far distance form node i may lead to a significant change in the coreness of i.

Having large networks in mind, we address the following issue: a node cannot have full information about the structure of the whole network, neither it can learn it in a reasonable time. Since ignoring the network structure may result in misleading predictions, we propose the following Bayesian setting to model such network games: nature chooses the degree of each player; the degree distribution is common knowledge to all nodes. In this setting, each player chooses the action that maximizes its expected payoff. We analyze the largest symmetric Baysian Nash equilibrium of this game. Moreover, we establish the monotonicity of the largest BNE in the edge perspective degree distribution, under the assumption that neighbors' degrees are asymptotically independent.

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