Superposition Encoding and Partial Decoding Is Optimal for a Class of Z-interference Channels

Nan Liu and Andrea Goldsmith
Department of Electrical Engineering
Stanford University, Stanford CA 94305
Email: nanliu@stanford.edu andrea@wsl.stanford.edu

***Abstract*— We apply a technique introduced by Korner and Marton to the converse of a class of Z-interference channels. This class has the properties that, for the transmitter-receiver pair that suffers from interference, 1) the conditional output entropy is invariant with respect to the input and, 2) the maximum output entropy is achieved by a single input distribution irrespective of the interference distribution. We show that for this class of channels, superposition encoding and partial decoding is optimal. We thus provide a single-letter characterization for the capacity region, which was previously unknown.**

I. Introduction

The interference channel, introduced in [1], is a simple network consisting of two pairs of transmitters and receivers. Each pair wishes to communicate at a certain rate with negligible probability of error. However, the two communications interfere with each other. The problem of finding the capacity region of the interference channel is difficult and therefore remains essentially open except in some special cases, e.g., interference channels with statistically equivalent outputs [2]–[4], discrete additive degraded interference channels [5], a class of deterministic interference channels [6], strong interference channels [7]–[11], and a class of degraded interference channels [12].

The Z-interference channel is an interference channel where one transmitter-receiver pair is interference-free. Though finding the capacity region of the Z-interference channel is a simpler problem than that of the interference channel, capacity results are still limited, with the following exceptions: Z-interference channel where the interference is deterministic [6, Section IV], the sum capacity of the Gaussian Z-interference channel [13], and the Z-interference channel where the interference-free link is also noise-free [14].

The largest achievable region of the interference channel is given by Han and Kobayashi in 1981 [9], which uses the idea of superposition encoding and partial decoding. This achievability scheme requires the transmitters to encode their messages via superposition encoding, and each receiver to decode not only its own message, but also part of the interference. One of the main difficulties in finding the capacity region of the interference channel is to justify the need for partial decoding in the converse proofs, as the partially decoded information is not required at the receiver. So far, the only result that proves superposition encoding and partial decoding is optimal is [6] where, due to the deterministic nature of the channel, the converse results obtained by using the technique of genie-aided receivers is sufficient to meet the achievability results of [9].

In this paper, we use a different converse technique and prove that superposition encoding and partial decoding is optimal for a class of Z-interference channels that are not necessarily deterministic. The technique [15, page 314] that we use in obtaining the converse was introduced by Korner and Marton in [16], which has been useful in the solution of several problems in multi-user information theory, e.g., broadcast channels with degraded message sets [17], images of a set via two channels [16], communication where the transmitter has non-causal perfect side information, i.e., the Gelfand-Pinsker problem [18], and semicodes for the multiple access channel [14]. We apply this technique to the converse of a class of Z-interference channels. As a result, we show that superposition encoding and partial decoding is optimal for a class of Z-interference channels and, therefore, provide a single-letter characterization for the capacity region of this class, which was previously unknown.

II. System Model

Consider a Z-interference channel with two transition probabilities $p(y_1|x_1)$ and $p(y_2|x_1,x_2)$. The input and output alphabets are $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Set

$$V_1(a|b) = \Pr[Y_1 = a|X_1 = b], \tag{1}$$

$$V_2(c|b,d) = \Pr[Y_2 = c|X_1 = b, X_2 = d]. \tag{2}$$

Let W_1 and W_2 be two independent messages uniformly distributed on $\{1,2,\cdots,M_1\}$ and $\{1,2,\cdots,M_2\}$, respectively. Transmitter i wishes to send message W_i to receiver i, $i=1,2$. An (M_1,M_2,n,ϵ_n) code for this channel consists of a sequence of two encoding functions

$$f_i^n : \{1,2,\cdots,2^{nR_i}\} \to \mathcal{X}_i^n, \qquad i=1,2 \tag{3}$$

and two decoding functions

$$g_i^n : \mathcal{Y}_i^n \to \{1,2,\cdots,2^{nR_i}\}, \qquad i=1,2 \tag{4}$$

with probability of error ϵ_n defined as

$$\max_{i=1,2} \frac{1}{M_1M_2} \sum_{w_1,w_2} \Pr\left[g_i^n(Y_i^n) \neq w_i | W_1 = w_1, W_2 = w_2\right].$$

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A rate pair (R_1, R_2) is said to be achievable if there exists a sequence of $\left(2^{nR_1}, 2^{nR_2}, n, \epsilon_n\right)$ codes such that $\epsilon_n \to 0$ as $n \to \infty$. The capacity region of the Z-interference channel is the closure of the set of all achievable rate pairs.

An example of the Z-interference channel is the Gaussian Z-interference channel, where $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{R}$, and $p(y_1|x_1)$ and $p(y_2|x_1, x_2)$ are given as

$$Y_1 = X_1 + Z_1, \tag{5}$$

$$Y_2 = aX_1 + X_2 + Z_2, \tag{6}$$

where Z_1 and Z_2 are independent Gaussian random variables with zero mean and unit variance, $a \in \mathbb{R}$, and the channel inputs have to satisfy the average power constraints of P_1 and P_2.

The class of Z-interference channels we investigate in this paper satisfies the following conditions:

1) For any $n = 1, 2, \cdots$, $H(Y_2^n|X_2^n = x_2^n)$ is independent of x_2^n for any $p(x_1^n)$.
2) Define τ as

$$\tau = \max_{p(x_1)p(x_2)} H(Y_2). \tag{7}$$

Then there exists a $p^*(x_2)$ such that $H(Y_2)$, when evaluated with the distribution $\sum_{x_1,x_2} p(x_1)p^*(x_2)$ $p(y_2|x_1, x_2)$, is equal to τ for any $p(x_1)$.

The first condition specifies that the channel $p(y_2|x_1, x_2)$ is invariant, in terms of conditional output entropy, with respect to the input sequence of transmitter 2, i.e., x_2^n. This means that when designing the codebook of transmitter 2, the exact locations of the codewords do not affect the performance, rather, it is the relative locations, or "distances", between codewords that matter. For example, the Gaussian Z-interference channel, defined in (5) and (6), satisfies this condition.

Intuitively, the second condition specifies that no matter how tightly packed the codewords in codebook 1 are, by spacing out the codewords in codebook 2, we can always fill up the entire, or maximum, output space of receiver 2. This means that using an i.i.d. generated codebook with $p^*(x_2)$ at transmitter 2 is to our advantage, as the larger the output space, the more codewords of transmitter 2 we can pack in the space. Note that the Gaussian Z-interference channel does not satisfy this condition, since the largest output space is only achieved when both $p(x_1)$ and $p(x_2)$ are Gaussian with variances P_1 and P_2, respectively. The largest output space cannot be achieved with a $p^*(x_2)$ that is irrespective of $p(x_1)$, as specified in condition 2.

Now, we give an example of a channel, shown in Figure 1, where both conditions are satisfied. Let $\mathcal{X}_2 = \mathcal{Y}_2 = \mathcal{S} = \{0, 1, 2, \cdots, q-1\}$, where q is an arbitrary interger. Let sets $\mathcal{X}_1$, $\mathcal{Y}_1$, and probability distributions $\bar{p}(y_1|x_1)$ and $\bar{p}(s|x_1)$ be arbitrary. The channel is defined as $V_1(y_1|x_1) = \bar{p}(y_1|x_1)$, and $V_2(y_2|x_1, x_2)$ is defined as $\sum_{s \in \mathcal{S}} \bar{p}(y_2|s, x_2)\bar{p}(s|x_1)$, where $\bar{p}(y_2|s, x_2)$ is given by

$$Y_2 = S \oplus X_2. \tag{8}$$

In (8), $\oplus$ is the mod-q sum. It is easy to see that the channel

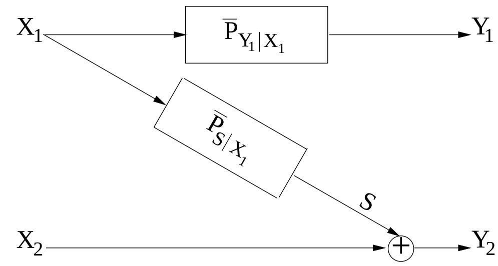

Fig. 1. Example of a Z-interference channel satisfying conditions 1 and 2

thus defined satisfies condition 1. By letting $p^*(x_2)$ be the uniform distribution on $\{0, 1, 2, \cdots, q-1\}$, we find that the channel also satisfies condition 2, where $\tau = \log |\mathcal{Y}_2| = \log q$.

The above example is related to several Z-interference channels that have been studied in [5], [6], [19].

1. The discrete additive degraded interference channels studied in [5], using similar techniques as [20, Fig. 6], can be shown to be equivalent to, or in other words, have the same capacity region as, the following Z-inteference channel:

$$Y_1 = X_1 \oplus Z_1 \tag{9}$$

$$Y_2 = X_1 \oplus X_2 \oplus Z_1 \oplus Z_2. \tag{10}$$

The Z-interference channel characterized by (9) and (10) is a special case of our example. The derivation of the capacity region in [5] relies on the degradedness of output Y_2 with respect to Y_1, which makes treating interference as noise optimal. In our example, we do not make an assumption on degradedness, and show that superposition encoding and partial decoding is optimal.

2. The example shown in [6, Fig. 3] is a special case of our example, with $\mathcal{X}_2 = \mathcal{Y}_2 = \{0, 1\} = \mathcal{S} = \mathcal{Y}_1$ and $\mathcal{X}_1 = \{0, 1, 2\}$, $\bar{p}(y_1|x_1)$ is given as

$$\bar{p}(y_1|x_1) = \begin{cases} 1 & \text{if } (x_1, y_1) = (0,0) \text{ or } (x_1, y_1) = (1,1) \\ \epsilon & \text{if } (x_1, y_1) = (2,0) \\ 1-\epsilon & \text{if } (x_1, y_1) = (2,1) \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

and $\bar{p}(s|x_1)$ is given as

$$\bar{p}(s|x_1) = \begin{cases} 1 & \text{if } (x_1, s) = (0,0) \text{ or } (x_1, s) = (1,1) \\ & \text{or } (x_1, s) = (2,0) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

which is deterministic. As mentioned before, the capacity region of this channel is found in [6] by matching the achievability of superposition encoding and partial decoding and the converse of genie-aided receivers. The deterministic nature of $p(y_2|x_1, x_2)$ plays an important role in obtaining the results. In our example, we make no assumption on the channel being deterministic, and use a different converse technique to match the achievability of superposition encoding and partial decoding.

3. For a class of interference channels, [19] quantifies the gap between the achievability of superposition encoding and partial

decoding and the converse obtained using the technique of genie-aided receivers. Consider a special case of the class of interference channels considered in [19], where $p(s_2|x_2)$ in [19, Fig. 1] is independent of x_2 and the deterministic function f_2 in [19, Fig. 1] is the modulo sum operation. This special case is contained in our example. Hence, for this special case, the results in this paper provide the exact capacity region, and we may conclude that the achievable region of Han and Kobayashi used in [19] is in fact optimal, while the converse in [19] is not tight.

III. Converse

In this section, we prove a converse result for the class of Z-interference channels that satisfy condition 1.

Before we proceed, we will first restate the converse technique introduced in Korner and Marton [16]. The technique is a method of writing the difference between two n-letter entropies into a sum of differences between conditional entropies, which may then be written as the difference between two single letter conditional entropies by defining the appropriate auxiliary random variables.

Lemma 1: [15, page 314, eqn (3.34)]

For any n, and any random variables Y^n and Z^n, we have

$$H(Z^n)-H(Y^n) = \sum_{i=1}^{n} \big(H(Z_i|Y^{i-1}, Z_{i+1}, Z_{i+2}, \cdots, Z_n) - H(Y_i|Y^{i-1}, Z_{i+1}, Z_{i+2}, \cdots, Z_n) \big). \quad (13)$$

Now, we state our converse result for the class of Z-interference channels that satisfy condition 1.

Theorem 1: For a Z-interference channel, characterized by transition probabilities V_1 and V_2, that satisfies condition 1, if rate pair (R_1, R_2) is achievable, then it must satisfy

$$R_1 \leq H(Y_1|U) + \gamma - H(Y_1|X_1) \quad (14)$$
$$R_2 \leq \tau - H(T|U) - \gamma \quad (15)$$
$$0 \leq \gamma \leq \min(I(Y_1;U), I(T;U)) \quad (16)$$

for some distributions $p(u)p(x_1|u)$ and number γ, where the mutual informations and entropies are evaluated using $p(u, x_1, y_1, t) = p(u)p(x_1|u)V_1(y_1|x_1)V_2(t|x_1, \bar{x}_2)$, and $\bar{x}_2$ is an arbitrary element in $\mathcal{X}_2$.

Proof: Since the rate pair (R_1, R_2) is achievable, there exist two sequences of codebooks 1 and 2, denoted by $\mathcal{C}_1^n$ and $\mathcal{C}_2^n$, of rate R_1 and R_2, and probability of error less than ϵ_n, where $\epsilon_n \to 0$ as $n \to \infty$. Let X_1^n and X_2^n be uniformly distributed on codebooks 1 and 2, respectively. Let Y_1^n be connected via V_1^n to X_1^n, Y_2^n be connected via V_2^n to X_1^n and X_2^n.

We start the converse with Fano's inequality [21] and the data processing inequality [21],

$$\begin{aligned} nR_1 = H(W_1) &= I(W_1;Y_1^n) + H(W_1|Y_1^n) && (17)\\ &\leq I(W_1;Y_1^n) + n\epsilon_n && (18)\\ &\leq I(X_1^n;Y_1^n) + n\epsilon_n && (19)\\ &= H(Y_1^n) - H(Y_1^n|X_1^n) + n\epsilon_n && (20)\\ &= H(Y_1^n) - \sum_{i=1}^{n} H(Y_{1i}|X_{1i}) + n\epsilon_n && (21)\end{aligned}$$

and (21) follows from the memoryless nature of V_1^n. We also have

$$\begin{aligned} nR_2 = H(W_2) &= I(W_2;Y_2^n) + H(W_2|Y_2^n) && (22)\\ &\leq I(W_2;Y_2^n) + n\epsilon_n && (23)\\ &\leq I(X_2^n;Y_2^n) + n\epsilon_n && (24)\\ &= H(Y_2^n) - H(Y_2^n|X_2^n) + n\epsilon_n && (25)\\ &\leq \sum_{i=1}^{n} H(Y_{2i}) - H(Y_2^n|X_2^n) + n\epsilon_n && (26)\\ &\leq n\tau - H(Y_2^n|X_2^n) + n\epsilon_n && (27)\end{aligned}$$

where (26) follows because conditioning reduces entropy, and (27) follows from the definition of τ in (7).

Let us define another channel, $\hat{V}_2 : \mathcal{X}_1 \to \mathcal{Y}_2$, as

$$\hat{V}_2(t|x_1) = V_2(t|x_1, \bar{x}_2), \quad (28)$$

where $\bar{x}_2$ is an arbitrary element in $\mathcal{X}_2$. Further, let us define another sequence of random variables, T^n, which is connected via $\hat{V}_2^n$, the memoryless channel $\hat{V}_2$ used n times, to X_1^n, i.e., $T_i \to X_{1i} \to T_{\{i\}^c}, X_{1\{i\}^c}, X_2^n, Y_1^n, Y_2^n$. Also define $\bar{x}_2^n$ as the n-sequence with $\bar{x}_2$ repeated n times. It is easy to see that

$$\begin{aligned} H(Y_2^n|X_2^n) &= \sum_{x_2^n \in \mathcal{C}_2^n} \frac{1}{2^{nR_2}} H(Y_2^n|X_2^n = x_2^n) && (29)\\ &= H(Y_2^n|X_2^n = \bar{x}_2^n) && (30)\\ &= H(T^n), && (31)\end{aligned}$$

where (30) follows from the fact that the channel under consideration satisfies condition 1, and (31) follows from the definition of T^n.

By applying Lemma 1, we have

$$H(T^n)-H(Y_1^n) = \sum_{i=1}^{n} \big(H(T_i|Y_1^{i-1}, T_{i+1}, T_{i+2}, \cdots, T_n) - H(Y_{1i}|Y_1^{i-1}, T_{i+1}, T_{i+2}, \cdots, T_n) \big). \quad (32)$$

Furthermore, since conditioning reduces entropy, we can write

$$H(Y_1^n) = \sum_{i=1}^{n} H(Y_{1i}|Y_1^{i-1}) \leq \sum_{i=1}^{n} H(Y_{1i}) \quad (33)$$

$$H(T^n) = \sum_{i=1}^{n} H(T_i|T^{i-1}) \leq \sum_{i=1}^{n} H(T_i) \quad (34)$$

$$\begin{aligned} H(Y_1^n) &= \sum_{i=1}^{n} H(Y_{1i}|Y_1^{i-1})\\ &\geq \sum_{i=1}^{n} H(Y_{1i}|Y_1^{i-1}, T_{i+1}, T_{i+2}, \cdots, T_n). && (35)\end{aligned}$$

Define the following auxiliary random variables,

$$U_i = Y_1^{i-1}, T_{i+1}, T_{i+2}, \cdots, T_n, \qquad i = 1, 2, \cdots, n \quad (36)$$

Hence, we have

$$H(T^n) - H(Y_1^n) = \sum_{i=1}^{n} \left(H(T_i|U_i) - H(Y_{1i}|U_i) \right) \quad (37)$$

$$\sum_{i=1}^{n} H(Y_{1i}|U_i) \le H(Y_1^n) \le \sum_{i=1}^{n} H(Y_{1i}) \quad (38)$$

$$H(T^n) \le \sum_{i=1}^{n} H(T_i). \quad (39)$$

Further define Q as a random variable that is uniform on the set $\{1, 2, \cdots, n\}$ and independent of everything else. Also, define the following auxiliary random variables:

$$U = (U_Q, Q), \quad X_1 = X_{1Q}, \quad Y_1 = Y_{1Q}, \quad T = T_Q. \quad (40)$$

Then, we have

$$n^{-1} \left(H(T^n) - H(Y_1^n) \right) = H(T|U) - H(Y_1|U) \quad (41)$$

$$H(Y_1|U) \le n^{-1} H(Y_1^n) \le H(Y_1|Q) \le H(Y_1) \quad (42)$$

$$n^{-1} H(T^n) \le H(T|Q) \le H(T). \quad (43)$$

Due to the memoryless nature of V_1^n and $\hat{V}_2^n$, the fact that Q is independent of everything else, and the Markov chain $T_i \to X_{1i} \to Y_{1i}$, for $i = 1, 2, \cdots, n$, the joint distribution of U, X_1, Y_1, T satisfies

$$p(u, x_1, y_1, t) = p(u) p(x_1|u) V_1(y_1|x_1) V_2(t|x_1, \bar{x}_2). \quad (44)$$

From (41)-(43), we may conclude that there exists a number γ that satisfies

$$0 \le \gamma \le \min(I(Y_1; U), I(T; U)) \quad (45)$$

such that

$$\frac{1}{n} H(T^n) = H(T|U) + \gamma, \quad \frac{1}{n} H(Y_1^n) = H(Y_1|U) + \gamma. \quad (46)$$

From the memoryless nature of V_1^n, we see that

$$\Pr[Y_1 = y_1 | X_1 = x_1, Q = i] = V_1(y_1|x_1), \quad (47)$$

which means

$$H(Y_1|X_1) = H(Y_1|X_1, Q) = \frac{1}{n} \sum_{i=1}^{n} H(Y_i|X_i). \quad (48)$$

By combining (21), (27), (31), (45), (46) and (48), and allowing n to approach infinity, we obtain the desired result. ■

IV. Achievability

In this section, we state an achievability result, based on [9], that is true for any Z-interference channel. Since there are no new concepts introduced in the achievability scheme, we merely state the result without going into the details of the calculation of error probability.

The achievability scheme is described as follows.

Codebook generation: Fix some distribution $p(u)p(x_1|u)$ $p(x_2)$. The technique of superposition encoding is used at transmitter 1. More specifically, for the codebook at transmitter 1, we first generate an inner codebook of $2^{n\gamma}$ rows in an i.i.d. fashion using $p(u)$. Then, conditioned on each row of the inner codebook, we generate an outer codebook of $2^{n(R_1-\gamma)}$ rows in an i.i.d. fashion using $p(x_1|u)$. For the codebook at transmitter 2, we generate a codebook of 2^{nR_2} rows in an i.i.d. fashion using $p(x_2)$.

Encoding: The message at transmitter 1, W_1, contains R_1 bits. Transmitter 1 splits W_1 into two independent parts: W_c which contains γ bits, and W_p, which contains $R_1 - \gamma$ bits. Suppose $W_c = w_c$, where $w_c = 1, 2, \cdots, 2^{n\gamma}$, and $W_p = w_p$, where $w_p = 1, 2, \cdots, 2^{n(R_1-\gamma)}$. The transmitted codeword of transmitter 1 is the w_p-th row of the w_c-th outer codebook. Suppose the message at transmitter 2, W_2, is equal to w_2, where $w_2 = 1, 2, \cdots, 2^{nR_2}$. Transmitter 2 transmits the w_2-th row of its codebook.

Decoding and performance: Receiver 1 decodes the inner codeword first, while treating the outer codeword as noise. More specifically, it finds the inner codeword that is jointly typical with the received signal. After receiver 1 has identified the inner codeword, it decodes the outer codeword. More specifically, it finds the outer codeword, which is in the outer codebook of the inner codeword, that is jointly typical with the received signal. Receiver 2 decodes both the inner codeword of transmitter 1 and the codeword of transmitter 2 while treating the outer codeword of transmitter 1 as noise, as a receiver in a multiple access channel. More specifically, it finds the pair consisting of the inner codeword of transmitter 1 and the codeword of transmitter 2 that is jointly typical with the received signal. The rates achievable using the described achievability scheme is given in the next theorem.

Theorem 2: For a Z-interference channel, characterized by transition probabilities V_1 and V_2, if (R_1, R_2) satisfies

$$R_1 \le I(X_1; Y_1|U) + \gamma \quad (49)$$

$$R_2 \le \min(I(X_2; Y_2|U), I(U, X_2; Y_2) - \gamma) \quad (50)$$

$$0 \le \gamma \le \min(I(U; Y_1), I(U; Y_2|X_2)) \quad (51)$$

for some distributions $p(u)p(x_1|u)p(x_2)$ and number γ, where the mutual informations are evaluated using $p(u, x_1, x_2, y_1, y_2) = p(u)p(x_1|u)p(x_2)V_1(y_1|x_1)V_2(y_2|x_1, x_2)$, then rate pair (R_1, R_2) is achievable.

V. Capacity Region

In general, for Z-interference channels that satisfy condition 1, the converse result, described in Theorem 1, and the achievability result, described in Theorem 2, do not meet. However, in this section, we show that when the Z-interference channel further satisfies condition 2, the converse result and the achievable region coincide, yielding the capacity region.

In the achievability result in Theorem 2, we specify $p(x_2)$ to be $p^*(x_2)$. Since the Z-interference channel satisfies condition 2, we have

$$H(Y_2|U) = \sum_{u} p(u) H(Y_2|U = u) = \tau = H(Y_2). \quad (52)$$

Therefore, we obtain another achievable region, potentially smaller than that described in Theorem 2 due to the fact that we fixed $p(x_2)$ to be a specific distribution:

$$R_1 \leq H(Y_1|U) - H(Y_1|X_1) + \gamma \tag{53}$$
$$R_2 \leq \tau - H(Y_2|X_2, U) - \gamma \tag{54}$$
$$0 \leq \gamma \leq \min(I(U;Y_1), I(U;Y_2|X_2)) \tag{55}$$

for some $p(u)p(x_1|u)$, where the mutual informations and entropies are evaluated with $p(u,x_1,x_2,y_1,y_2) = p(u)p(x_1|u)p^*(x_2)V_1(y_1|x_1)V_2(y_2|x_1,x_2)$.

Next, we will show that when the Z-interference channel satisfies conditions 1 and 2, the region described by (14)-(16) and that described by (53)-(55) are the same when evaluated with the same $p(u)p(x_1|u)$.

When evaluated with the same $p(u)p(x_1|u)$, it is obvious that the terms $H(Y_1|U)$, $H(Y_1|X_1)$ and $I(U;Y_1)$ are equal in both (14)-(16) and (53)-(55). $H(T|U)$, respectively $H(T)$, evaluated with the converse distributions is equal to $H(Y_2|X_2 = \bar{x}_2, U)$, respectively $H(Y_2|X_2 = \bar{x}_2)$, evaluated with the achievability distributions. Furthermore, evaluated with the achievability distributions,

$$H(Y_2|X_2, U)$$
$$= \sum_{x_2,u} p^*(x_2)p(u)H(Y_2|X_2 = x_2, U = u) \tag{56}$$
$$= \sum_{x_2,u} p^*(x_2)p(u)H(Y_2|X_2 = \bar{x}_2, U = u) \tag{57}$$
$$= H(Y_2|X_2 = \bar{x}_2, U), \tag{58}$$

where we obtain (57) using condition 1 with $n = 1$. We also have

$$H(Y_2|X_2) = \sum_{x_2} p^*(x_2)H(Y_2|X_2 = x_2) \tag{59}$$
$$= H(Y_2|X_2 = \bar{x}_2). \tag{60}$$

Thus, we have proved that the region described by (14)-(16) and that described by (53)-(55) are the same when evaluated with the same $p(u)p(x_1|u)$.

Since the converse region and the achievability region are taking the union of regions described by (14)-(16) and (53)-(55), respectively, over all $p(u)p(x_1|u)$, the achievable region and the converse coincide, yielding the capacity region.

Hence, the capacity region of the class of Z-interference channels that satisfies conditions 1 and 2 is

$$R_1 \leq I(X_1;Y_1|U) + \min(I(U;Y_1), I(U;Y_2|X_2)) \tag{61}$$
$$R_2 \leq \tau - H(Y_2|X_2, U) \tag{62}$$
$$R_1 + R_2 \leq I(X_1;Y_1|U) + \tau - H(Y_2|X_2, U) \tag{63}$$

for some $p(u)p(x_1|u)$, where the mutual informations and entropies are evaluated with $p(u,x_1,x_2,y_1,y_2) = p(u)p(x_1|u)p^*(x_2)V_1(y_1|x_1)V_2(y_2|x_1,x_2)$. Using support lemma [15, Lemma 3.4], without loss of generality, we may bound the cardinality of the auxiliary random variable U as $|\mathcal{U}| \leq |\mathcal{X}_1| + 1$.

VI. Conclusion

We have determined the capacity region for a class of Z-interference channels by exploiting a converse technique from Korner and Marton. These results expand the set of interference channels with known capacity regions. Our results indicate that partial decoding is required to achieve capacity for certain Z-interference channels. The technique employed may also be of use in providing new converse results for other classes of interference channels in addition to the one considered in this paper.

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