Distortion Metrics of Composite Channels with Receiver Side Information

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***Abstract*— We consider transmission of stationary ergodic sources over non-ergodic composite channels with channel state information at the receiver (CSIR). Previously we introduced alternative capacity definitions to Shannon capacity, including outage and expected capacity. These generalized definitions relax the constraint of Shannon capacity that all transmitted information must be decoded at the receiver. In this work alternative end-to-end distortion metrics such as outage and expected distortion are introduced to relax the constraint that a single distortion level has to be maintained for all channel states. Through the example of transmission of a Gaussian source over a slow-fading Gaussian channel, we illustrate that the end-to-end distortion metrics dictate whether the source and channel coding can be separated for a communication system. We also show that the source and channel need to exchange information through an appropriate interface to facilitate separate encoding and decoding.**

I. Introduction

End-to-end distortion is a well-accepted metric for transmission of a stationary ergodic source over stationary ergodic channels. In this work we consider transmission of a stationary ergodic source over non-ergodic *composite channels*. A composite channel is a collection of channels $\{W_S : S \in \mathcal{S}\}$ parameterized by S, where the random variable S is chosen according to some distribution $p(S)$ at the beginning of transmission and then held fixed. We assume the channel realization is revealed to the receiver but not the transmitter.

The capacity of a composite channel is given by the Verdú-Han generalized capacity formula [1] as

$$C = \sup_{\boldsymbol{X}} \underline{\boldsymbol{I}}(\boldsymbol{X};\boldsymbol{Y}),$$

where $\underline{\boldsymbol{I}}(\boldsymbol{X};\boldsymbol{Y})$ is the liminf in probability of the normalized information densities. This formula highlights the pessimistic nature of the Shannon capacity definition – the capacity and consequently the end-to-end distortion are dominated by the performance of the "worst" channel, no matter how small its probability. To provide more flexibility in capacity definitions, in [2], [3] we relax the constraint that all transmitted information has to be correctly decoded and derive alternative capacity definitions including the outage and expected capacity. Previously examined in [4], outage capacity is a common criterion used in wireless fading channels. In [5] Shamai et al. also derives the expected capacity for a Gaussian slow-fading channel.

Similarly, in considering end-to-end distortion we can relax the constraint that a single distortion level has to be maintained for all channel states. Along these lines we introduce generalized end-to-end distortion metrics including the *outage distortion* and the *expected distortion*. The outage distortion is characterized by a pair (q, D_q), where the distortion level D_q is guaranteed with probability no less than $(1-q)$. The expected distortion is defined as $\mathbb{E}_S D_S$, where D_S is the achievable distortion when the channel is in state S and the expectation is taken with respect to the underlying distribution $p(S)$. These alternative distortion metrics are also considered in prior works. In [6] the overall distortion $q\sigma^2 + (1-q)D_q$, averaged over the non-outage and outage states, was adopted to analyze a two-hop fading channel. Here σ^2 is the variance of the source symbols. The expected distortion was analyzed for the MIMO block fading channel in the high SNR regime [7] and in the finite SNR regime [8], [9].

For transmission of a stationary ergodic source over a stationary ergodic channel, the separation theorem [10, Theorem 2.4] asserts that a target distortion level D is achievable if the channel capacity C exceeds the source rate distortion function $R(D)$, and a two-stage separate source-channel code suffices to meet the requirement. However, there are examples in multi-user channels [11] where the separation theorem fails. In this work we study whether the source-channel separation holds for generalized channel models and distortion metrics in point-to-point communications.

Source-channel separation can be defined in terms of code design. For transmission of a source over a channel the system consists of three concatenated blocks: the encoder f_n that maps the source symbols V^n to the channel input X^n; the channel W^n that maps the channel input X^n to channel output Y^n, and the decoder ϕ_n that maps the channel output Y^n to a reconstruction of source symbols $\hat{V}^n$. Source-channel separation dictates that the encoder f_n is separated into a source encoder

$$\hat{f}_n : V^n \to \{1, 2, \cdots, M_s\}$$

and a channel encoder

$$\tilde{f}_n : \{1, 2, \cdots, M_c\} \to X^n,$$

where $M_s \leq M_c$. Similarly the decoder ϕ_n is separated into a channel decoder $\tilde{\phi}_n$ and a source decoder $\hat{\phi}_n$. In contrast joint source-channel coding is a loose label that encompasses all coding techniques where the source and channel coders are not entirely separated. Consider as an example the *direct* *transmission* of a Gaussian source $\mathcal{CN}(0, \sigma^2)$ over a Gaussian channel with input power constraint P. The linear encoder $X = f(V) = \sqrt{P/\sigma^2}V$ cannot be separated into a source encoder and a channel encoder. Therefore this direct transmission is an example of joint-source channel coding.

Source-channel separation implies that the operation of source and channel coding does not depend on the statistics of the counterpart. However, the source and channel do need to communicate through an *interface*. In the classical example of station ergodic sources and channels, the source requires a rate $R(D)$ based on the target distortion D and the channel decides if it can support the rate based on its capacity C. For generalized source and channel models and distortion metrics, the interface is not necessarily a single rate and may allow multiple parameters to be agreed on between the source and channel. In [12] Vembu et al. studied the transmission of non-stationary sources over non-stationary channels. It is observed that the appropriate interface requires the source to provide the distribution of the *entropy density* and the channel to provide the distribution of the *information density* [12, Theorem 7]. From the source's point of view, there is a class of channels that have the same information density distribution, and the source can be transmitted over any channel within this class and be recovered at the receiver. The source is indifferent to the statistics of each individual channel and consequently the source coding does not depend on these statistics. We can argue similarly from the channel's standpoint.

In this work we consider the transmission of a Gaussian source over a slow-fading Gaussian channel and illustrate that the end-to-end distortion metrics dictate whether the source and channel coding can be separated for a communication system: separation holds under the outage distortion metric but fails under the expected distortion metric. We also show that the source and channel need to exchange information through an appropriate interface, which may not be a single rate, in order to facilitate separate source-channel coding.

The rest of the paper is organized as follows. We review alternative channel capacity definitions in Section II and define generalized end-to-end distortion metrics in Section III. In Section IV we study the transmission of a Gaussian source over a slow-fading Gaussian channel. We show that the end-to-end distortion metric dictates the separability of source and channel coding and also the appropriate source-channel interface. Conclusions are given in Section V.

II. Background: Channel Capacity Metrics

We review alternative channel capacity definitions derived in [2], [3] to provide some background information. A *composite channel* is a collection of ergodic stationary channels $\{W_S : S \in \mathcal{S}\}$ parameterized by S. The random variable S is chosen according to some distribution $p(S)$ at the beginning of the transmission and then held fixed. The realization of S is known at the receiver only and represented as an additional output. The conditional distribution from input to output is

$$P_{S,Y^n|X^n}(s, y^n|x^n) = P_S(s)P_{Y^n|X^n,S}(y^n|x^n, s). \quad (1)$$

The information density is defined similarly as in [1]

$$i(x^n; y^n|s) = \log \frac{P_{Y^n|X^n,S}(y^n|x^n, s)}{P_{Y^n|S}(y^n|s)}. \quad (2)$$

A. Outage Capacity

Consider a sequence of $(n, 2^{nR})$ codes. Let $P_o^{(n)}$ be the probability that the decoder declares an outage. Let $P_e^{(n)}$ be the probability that the receiver decodes improperly given that an outage is not declared. We say that a rate $(1-q)R$ is outage-q achievable if there exists a sequence of $(n, 2^{nR})$ channel codes such that $\lim_{n\to\infty} P_o^{(n)} \le q$ and $\lim_{n\to\infty} P_e^{(n)} = 0$. The outage-$q$ capacity of the above described channel is defined to be the supremum over all outage-q achievable rates, i.e. $C_q^o = (1-q)R_q$ where

$$R_q = \sup_{\boldsymbol{X}} \sup \left\{ \alpha : \lim_{n\to\infty} \Pr\left[\frac{1}{n} i(X^n; Y^n|S) \le \alpha\right] \le q \right\}. \quad (3)$$

In the case of a composite channel, the encoder uses a single code book and sends information at rate R_q. The receiver correctly decodes the information proportion $(1-q)$ of the time and declares an outage proportion q of the time. Thus the average rate is C_q^o. The value q can be chosen to maximize the outage capacity C_q^o.

B. Expected Capacity

Another strategy for increasing reliably-received rate is to use a single encoder at a rate R_t and a collection of decoders, each parameterized by s and decoding at a rate $R_s \le R_t$. The transmitter is forced to use a single encoder without channel side information, nevertheless the receiver can choose the appropriate decoder based on CSIR. Denote by $P_e^{(n,s)}$ the probability of error associated with channel s. We define the expected capacity C^e as the supremum of all achievable rates $\mathbb{E}_S R_S$ of any code sequence that satisfies $\mathbb{E}_S P_e^{(n,S)} \to 0$.

The expected capacity of the composite channel in (1) is closely related to the capacity region of a broadcast (BC) channel with $|\mathcal{S}|$ receivers, where we denote by $|\mathcal{S}|$ the cardinality of the user index set $\mathcal{S}$. In the broadcast system the channel from the input to the output of receiver s is

$$P_{Y_s^n|X^n}(y_s^n|x^n) = P_{Y^n|X^n,S}(y_s^n|x^n, s).$$

The capacity region $\mathcal{C}_{\text{BC}}$ of the above broadcast channel with common information is defined similarly as in [13, page 421]. We have message sets $\mathcal{W}_p = \{1, 2, \cdots, 2^{nR_p}\}$, where $p \subseteq \mathcal{S}$ is a non-empty subset of $\mathcal{S}$. The transmitted codeword is based on the Cartesian product of all messages $\prod_{p\subseteq\mathcal{S}} W_p$. The user with index s, upon receiving the channel output Y_s^n, will decode these messages W_p where $s \in p$. The BC capacity region consists of rate vectors (R_p) such that the decoding error vanishes with increasing block length. The following result in [3] relates the expected capacity of a composite channel to the capacity region of the corresponding BC:

$$C^e = \sup_{(R_p)\in\mathcal{C}_{\text{BC}}} \sum_{p\subseteq\mathcal{S}} R_p \sum_{s\in p} P(s) = \sup_{(R_p)\in\mathcal{C}_{\text{BC}}} \sum_{s\in\mathcal{S}} P(s) \sum_{s\in p} R_p, \quad (4)$$

where the achievable rate $R_s = \sum_{s\in p} R_p$ for channel state s.

III. End-to-End Distortion Metrics

We consider an ergodic stationary source that produces source symbols $V_1, V_2, \cdots, V_n$ drawn i.i.d. from a distribution $P(V)$. The source is transmitted over a composite channel $W^n : X^n \to (Y^n, S)$ with conditional output distribution

$$W^n(y^n, s|x^n) = P_S(s) P_{Y^n|X^n,S}(y^n|x^n, s).$$

Note that source and channel encoders, whether joint or separate, do not have access to channel state information S.

A. Outage Distortion

The objective is to achieve a distortion D_q with outage probability q. More specifically, we want to design an encoder $f_n : V^n \to X^n$ that maps the source symbols to the channel input and a decoder $\phi_n : (Y^n, S) \to \hat{V}^n$ that maps the channel output to an estimation of source symbols such that

$$\Pr\left\{(V^n, \hat{V}^n) : d(V^n, \hat{V}^n) \leq D_q\right\} \geq 1 - q, \tag{5}$$

where $d(V^n, \hat{V}^n) = \frac{1}{n}\sum_{i=1}^{n} d(V_i, \hat{V}_i)$ is the distortion measure between the source sequence V^n and its reconstruction $\hat{V}^n$. In order to evaluate (5) we need the conditional distribution $P(\hat{V}^n|V^n)$. Assuming the encoder f_n and the decoder ϕ_n are deterministic, this distribution is given by

$$\sum_{(X^n,Y^n,S)} W^n(Y^n, S|X^n) \cdot \mathbf{1}\left\{X^n = f_n(V^n), \hat{V}^n = \phi_n(Y^n, S)\right\} \tag{6}$$

Here $\mathbf{1}\{\cdot\}$ is the indicator function. Note that the channel statistics W^n and the source statistics $P(V^n)$ are fixed, so the code design is essentially the appropriate choice of the encoder-decoder pair (f_n, ϕ_n).

B. Expected Distortion

For the expected distortion metric, our design objective now changes from (5) to

$$\mathbb{E}_{(V^n,\hat{V}^n)}\left\{d(V^n, \hat{V}^n)\right\} \leq D^e, \tag{7}$$

where D^e is the target expected distortion. Using the conditional distribution $P(\hat{V}^n|V^n)$ in (6), the expected distortion can be rewritten as

$$\mathbb{E}_{(V^n,\hat{V}^n)}\left\{d(V^n, \hat{V}^n)\right\} = \mathbb{E}_S D_S = \sum_S P(S) D_S.$$

Here we denote by D_S the achievable average distortion when the channel is in state S, and it is given by

$$D_S = \sum P(V^n) W^n(Y^n|X^n, S) d(V^n, \hat{V}^n),$$

where the summation is over all $(V^n, X^n, Y^n, \hat{V}^n)$ such that $X^n = f_n(V^n)$ and $\hat{V}^n = \phi_n(Y^n, S)$.

Notice that when a stationary ergodic source is transmitted over a stationary ergodic channel, we can design source-channel codes such that $d(V^n, \hat{V}^n)$ approaches the same limit as $n \to \infty$. However, in the case of a composite channel it is possible that $d(V^n, \hat{V}^n)$ approaches different limits depending on the channel state S, so the expected distortion metric captures the distortion averaged over various channel states.

IV. Source-Channel Coding

In this section we consider transmission of a stationary ergodic source over non-ergodic composite channels. Through the example of transmission of a Gaussian source over a slow-fading Gaussian channel, we illustrate that the end-to-end distortion metrics dictate whether the source and channel coding can be separated. We also show that the source and channel need to exchange information through an appropriate interface to facilitate separate encoding and decoding.

We recall the definition of a source rate-distortion function as [14, page 342]

$$R(D) = \min_{P(\hat{V}|V):\mathbb{E}d(V,\hat{V}) \leq D} I(V; \hat{V}). \tag{8}$$

For a stationary ergodic source and channel, it is shown that if $R(D) < C$ then the source can be transmitted over the channel subject to an *average fidelity criterion* $\mathbb{E}\left\{d(V^n, \hat{V}^n)\right\} \leq D$. Conversely, if the transmission satisfies the average fidelity criterion, we also conclude $R(D) \leq C$ [10, page 130]. Next we consider composite channel models and generalized distortion metrics.

A. Source Channel Coding under an Outage Distortion Metric

Lemma IV.1 *The source can be transmitted over the channel and satisfy the outage distortion constraint* (5) *if*

$$R(D_q) < R_q = C_q^o/(1 - q),$$

where C_q^o is the outage capacity, R_q is defined in (3) *and $R(D_q)$ is the source rate distortion function* (8) *evaluated at distortion level D_q.*

This lemma gives a sufficient condition for the source to be transmitted over the channel subject to the outage distortion constraint (5). In the proof we see the design of encoder f_n and decoder ϕ_n involves a two-stage procedure, i.e the encoder f_n consists of a source encoder $\hat{f}_n$ and a channel encoder $\tilde{f}_n$, and similarly for the decoder ϕ_n. In fact Lemma IV.1 can be viewed as the direct part of source-channel separation under the outage distortion metric.

In the rate distortion theory for source coding, one often imposes the average fidelity criterion

$$\mathbb{E}\left\{d(V^n, \hat{V}^n)\right\} \leq D. \tag{9}$$

The main challenge here is to satisfy the condition (5) which is based on the tail of the distortion distribution rather than on its mean. So for source coding, instead of the global average fidelity criterion (9), we impose the following local *ϵ-fidelity criterion* [10, page 123]

$$\Pr\left\{(V^n, \hat{V}^n) : d(V^n, \hat{V}^n) \leq D\right\} \geq 1 - \epsilon. \tag{10}$$

In the limit of ϵ approaching 0, the ϵ-fidelity criterion (10) is a stronger condition than the average fidelity criterion (9). But for a fixed $\epsilon > 0$, neither of the criterions (9) and (10) implies the other. It is well known that for any $\delta > 0$ there exist source codes with rate $R < R(D) + \delta$ such that the average fidelity criterion is satisfied [14, page 351]. In order to prove Lemma IV.1, we need a stronger result as given by the following lemma [10, page 125]:

Lemma IV.2 *For any $0 < \epsilon < 1$ there also exist source codes with rate $R < R(D) + \delta$ such that the ϵ-fidelity criterion* (10) *is satisfied.*

The existence of these codes is essential to the following proof of Lemma IV.1.

Proof: In the following we denote $R = R(D_q)$ and $C = R_q = C_q^o/(1-q)$ to simplify notation. The outage distortion metric (5) is satisfied through a two-stage separate source-channel encoder and decoder. By Lemma IV.2, for any $0 < \epsilon < 1$ and $\delta > 0$, there exists source encoder

$$\hat{f}_n : V^n \to U \in \{1, 2, \cdots, 2^{n(R+\delta)}\}$$

and source decoder

$$\hat{\phi}_n : \hat{U} \in \{1, 2, \cdots, 2^{n(R+\delta)}\} \to \hat{V}^n$$

such that

$$\Pr\left\{d(V^n, \hat{V}^n) \le D\right\} \ge 1 - \epsilon.$$

By definition of $C = C_q^o/(1-q)$ there exist channel codes with channel encoder

$$\tilde{f}_n : U \in \{1, 2, \cdots, 2^{n(C-\delta)}\} \to X^n$$

and channel decoder

$$\tilde{\phi}_n : (Y^n, S) \to \hat{U} \in \{1, 2, \cdots, 2^{n(C-\delta)}\}$$

such that $\lim_{n\to\infty} P_o^{(n)} \le q$ and $\lim_{n\to\infty} P_e^{(n)} = 0$. For sufficiently small δ we have $R + \delta < C - \delta$, which guarantees the output of the source encoder $\hat{f}_n$ always lies in the domain of the channel encoder $\tilde{f}_n$. Now

$$\begin{aligned} & \Pr\left\{d(V^n, \hat{V}^n) \le D\right\} \\ \ge\ & \Pr\left\{d(V^n, \hat{V}^n) \le D, U = \hat{U}\right\} \\ \ge\ & (1 - P_o^{(n)})(1 - P_e^{(n)})(1 - \epsilon) \to 1 - q \end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$. ∎

We illustrate the separate source and channel codes constructed in the proof of Lemma IV.1 by the following example. As shown in Figure 1, a Gaussian source $\mathcal{CN}(0, \sigma^2)$ is transmitted over a Rayleigh slow-fading Gaussian channel with fading distribution

$$p(\gamma) = (1/\bar{\gamma})\, e^{-\gamma/\bar{\gamma}},$$

where $\bar{\gamma}$ is the average channel power gain. The transmitter has a power constraint P. The additive Gaussian noise is i.i.d. and normalized to have unit variance. In this example we index each channel by the power gain γ, which has the same role as the previous channel index s. For an outage probability q the corresponding threshold of channel gain is $\gamma_q = -\bar{\gamma}\log(1-q)$, so in non-outage states the channel can support a rate of

$$R_q = \log(1 + P\gamma_q) = \log\left[1 - P\bar{\gamma}\log(1-q)\right]. \tag{11}$$

The rate distortion function of a complex Gaussian source is given by $R(D_q) = \log(\sigma^2/D_q)$. From Lemma IV.1 if

$$\sigma^2/D_q < 1 - P\bar{\gamma}\log(1-q), \tag{12}$$

then the outage distortion requirement (5) can be satisfied by concatenation of a source code at rate $R(D_q)$ and a channel code at rate R_q as given in (11).

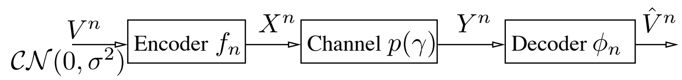

Fig. 1. Transmission of Gaussian source over slow-fading Gaussian channels

It is well known that the uncoded scheme is optimal for transmission of a Gaussian source over a Gaussian channel when the number of channel uses per source symbol is 1 [15]. The optimality is in the sense that a linear code $X = \sqrt{P/\sigma^2}V$ can achieve the minimum distortion

$$D_\gamma^* = \frac{\sigma^2}{1 + P\gamma} \tag{13}$$

for each channel state γ. It is easily seen that the optimal uncoded scheme also requires (12) in order to satisfy the outage distortion constraint.

We conclude that source-channel separation holds for this system under the outage distortion metric. The source-channel interface compares $R(D_q)$ and $R_q = C_q^o/(1-q)$ to determine whether the target distortion D_q is achievable with probability no less than $(1-q)$. If $R(D_q) < R_q$ then a separate source channel coding scheme suffices; if $R(D_q) > R_q$ then the outage distortion constraint can never be satisfied even for optimal joint source-channel coding.

For other systems that transmit ergodic stationary sources over composite channels, Lemma IV.1 gives the direct part of the source-channel separation under the outage distortion metric. A general converse is under investigation.

B. Source-Channel Separation Fails for Expected Distortion

Unlike the outage distortion metric, we do not believe that source-channel separation holds for the expected distortion metric. In the following we analyze the same example in Figure 1 under the expected distortion metric. We give the optimal expected distortion which is achievable with uncoded transmissions. We also analyze the achievable expected distortion under separate source-channel coding and characterize the corresponding distortion increase as compared to the optimal case.

1) Optimal Joint Source-Channel Coding: As aforementioned in Section IV-A, the uncoded scheme with a linear code $X = \sqrt{P/\sigma^2}V$ can achieve the minimum distortion (13) for each channel state γ, and therefore achieves the optimal expected distortion

$$(D^e)^* = \int_0^\infty \frac{\sigma^2 e^{-\gamma/\bar{\gamma}}}{1 + P\gamma} \cdot \frac{d\gamma}{\bar{\gamma}} = \frac{\sigma^2 e^{1/P\bar{\gamma}}}{P\bar{\gamma}} \mathrm{Ei}\left(\frac{1}{P\bar{\gamma}}\right), \tag{14}$$

where

$$\mathrm{Ei}(x) = \int_x^\infty \left(e^{-t}/t\right) dt$$

is the *exponential integral function*. Next we characterize the end-to-end distortion with separate source-channel coding.

2) Source-Channel Separation with Channel Codes for Outage Capacity: Consider using a channel code for outage capacity C_q^o and a source code at rate $R_q = C_q^o/(1-q)$ with R_q defined in (3). With probability q the channel is in outage so the receiver estimates the transmitted source symbol by its mean and the distortion is its variance σ^2. With probability $(1-q)$ the channel can support the rate R_q and the end-to-end distortion is $D_q = D(R_q)$. The overall expected distortion is averaged over the non-outage and outage states, i.e. $D_1^e(q) = q\sigma^2 + (1-q)D_q$. This fidelity criterion was also adopted in [6] to analyze a two-hop fading channel.

Under separate source-channel coding and channel codes for outage capacity, the minimum achievable distortion is obtained by optimizing $D_1^e(q)$ over $q \in (0,1)$. For the example in Figure 1 this becomes

$$D_1^e = \min_{0<q<1} D_1^e(q) = \min_{0<q<1} q\sigma^2 + \frac{(1-q)\sigma^2}{1 - P\bar{\gamma}\log(1-q)} \quad (15)$$

and the solution is to evaluate $D_1^e(q)$ at

$$q_D^* = 1 - \exp\left\{-\frac{2}{1+\sqrt{1+4P\bar{\gamma}}}\right\}. \quad (16)$$

One might be tempted to think that the channel should optimize its outage capacity

$$\max_{0<q<1} C_q^o = \max_{0<q<1} (1-q)\log\left[1 - P\bar{\gamma}\log(1-q)\right] \quad (17)$$

and provide $(q_C^*, R_{q_C^*})$ as the interface to the source, where q_C^* is the argument that maximizes (17). In fact the solution

$$q_C^* = 1 - \exp\left\{-\frac{e^{W(P\bar{\gamma})} - 1}{P\bar{\gamma}}\right\},$$

is in general different from q_D^* in (16), where $W(z)$ is the *Lambert-W* function satisfying $z = W(z)e^{W(z)}$. In case of separate source-channel coding where the channel has no access to the source statistics, it is insufficient for the channel to provide only $(q_C^*, R_{q_C^*})$ as the interface; instead it should provide the entire (q, R_q) curve and let the source choose the optimal operating point on this curve to minimize overall average distortion. Similarly, given a target expected distortion D, the source should determine for each outage probability q the corresponding outage distortion $D_q = (D - q\sigma^2)/(1-q)$ and provide the entire $(q, R(D_q))$ curve as the interface. With separate source and channel coding and a channel code for outage capacity, the expected distortion target is achievable if and only if there exists q such that $R(D_q) \le R_q$.

We illustrate the source-channel interface with a numerical example of the communication system in Figure 1: $\bar{\gamma} = 1$, $\sigma^2 = 1$ and $P = 10$. From (15) the minimum expected distortion $D_1^e = 0.443$ is obtained with $q_D^* = 0.237$. For three different expected distortion levels $D = \{0.9D_1^e,\ D_1^e,\ 1.1D_1^e\}$,

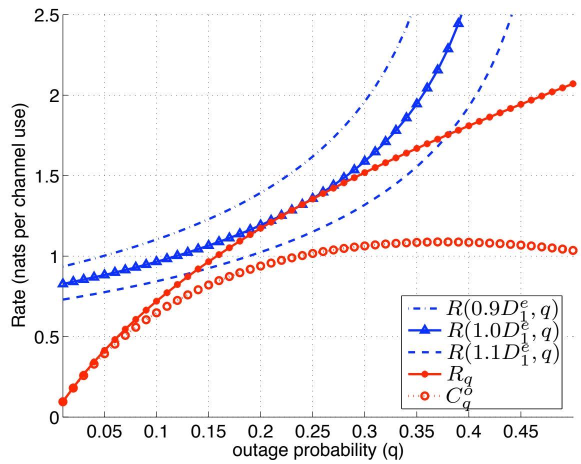

Fig. 2. Source channel interface with channel code for outage capacity

we compute for each outage probability q the corresponding outage distortion $D_q = (D - q\sigma^2)/(1-q)$ and the source coding rate $R(D_q)$. These curves are plotted in Figure 2 together with the outage capacity C_q^o and the rate R_q that can be supported by the channel for non-outage states. We observe the outage capacity is maximized at $q_C^* = 0.38 \neq q_D^*$, so in general we should compare the entire curve (q, R_q) and $(q, R(D_q))$ to determine whether the expected distortion target can be achieved with channel codes for outage capacity.

3) Source-Channel Separation with Channel Codes for Expected Capacity: We have seen in Section II that a composite channel can be viewed as a broadcast channel with virtual receivers indexed by each channel state. A broadcast channel code can be applied to achieve rate R_s when channel is in state s. It is well-known that a Gaussian source is successively refinable so we can design a multi-resolution source code which, when combined with the broadcast channel code, achieves a distortion $D(R_s)$ for each channel state s. The overall expected distortion is $\mathbb{E}_S D(R_S)$.

For the system under consideration, we assume a power allocation $\rho(\gamma) \ge 0$ which satisfies the overall power constraint $\int_0^\infty \rho(\gamma)d\gamma = P$. It is shown in [5] that the following rate is achievable

$$R(\gamma) = \int_0^\gamma \frac{u\rho(u)}{1 + uI(u)} du$$

when the channel gain is γ. Here $I(\gamma) = \int_\gamma^\infty \rho(u)du$ is the interference level when channel is in state γ. The minimum expected distortion with a multi-resolution source code and a broadcast channel code is then

$$\min_{\rho(\gamma)} \int_0^\infty \sigma^2 e^{-R(\gamma)} p(\gamma) d\gamma. \quad (18)$$

The solution to the above optimization problem is obtained in [9] as the limiting case of a discrete optimization. The optimal power allocation satisfies

$$\rho_D^*(\gamma) = \begin{cases} 0, & \gamma < \gamma_P \text{ or } \gamma > \bar{\gamma}, \\ -I'(\gamma), & \gamma_P \le \gamma \le \bar{\gamma}, \end{cases}$$

where

$$I(\gamma) = \frac{\int_{\bar{\gamma}}^{\gamma} \left(\frac{1}{2\bar{\gamma}} - \frac{1}{u}\right) e^{-u/2\bar{\gamma}} du}{\gamma e^{-\gamma/2\bar{\gamma}}},$$

and γ_P satisfies $I(\gamma_P) = P$. The corresponding minimum expected distortion is

$$D_2^e = \sigma^2 \left[D(\gamma_P) + \int_0^{\gamma_P} p(\gamma) d\gamma \right],$$

where

$$D(\gamma) = \frac{e^{-1} - \frac{1}{\bar{\gamma}} \int_{\bar{\gamma}}^{\gamma} e^{-(u+\bar{\gamma})/2\bar{\gamma}} \left(u/\bar{\gamma}\right)^{-1} du}{\left(\gamma/\bar{\gamma}\right)^{-1} e^{(\gamma-\bar{\gamma})/2\bar{\gamma}}}.$$

In general the optimal power allocation $\rho_C^*(\gamma)$ that maximizes the expected capacity $\int_0^\infty R(\gamma)p(\gamma)d\gamma$, as determined in [5], is different from $\rho_D^*(\gamma)$ that minimizes the expected distortion (18). Assuming separate source and channel coding and a broadcast channel code, the channel should provide the entire capacity region $\{(R_s)\}$ as the interface.

In Figure 3 we plot the expected distortion under the different source-channel coding schemes explored in this section. It is observed that the broadcast channel code combined with the multi-resolution source code performs slightly better than the channel code for outage capacity combined with a single rate source code, but there is a large gap between their expected distortion and that of the optimal uncoded scheme.

The uncoded transmission is essentially a joint source-channel coding scheme. Under the expected distortion metric, the source determines it is optimal to send uncoded symbols directly over the channel only after incorporating the statistics of a slow-fading Gaussian channel. The same uncoded transmission scheme may not be optimal or even inapplicable if the channel statistics change – consider instead we have a binary symmetric channel where the source output (complex values) does not match the channel input (binary values).

It is known that source-channel separation fails for certain multi-user channels [11]. Here we consider transmission of a Gaussian source over a slow-fading Gaussian channel and illustrate that even for point-to-point communication systems, under certain end-to-end distortion metrics such as expected distortion, separation also fails and joint source-channel coding is necessary to achieve the optimal performance.

V. Conclusion

We consider transmission of a stationary ergodic source over non-ergodic composite channels with channel state information at the receiver (CSIR). Similar to previously studied alternative channel capacity definitions such as outage and expected capacity, alternative end-to-end distortion metrics including outage and expected distortion are introduced in this work. We then study the transmission of a Gaussian source over slow-fading Gaussian channels and illustrate that the source-channel coding can be separated under an end-to-end outage distortion metric, while joint source-channel coding is optimal under an expected distortion metric. We also show that the source and channel need to exchange information through an appropriate interface in order to facilitate separate source-channel coding.

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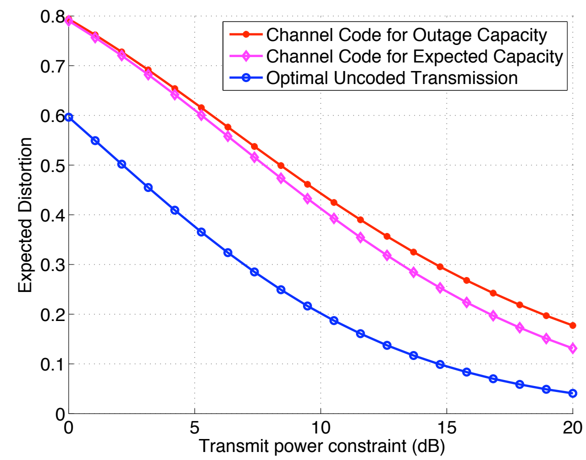

Fig. 3. Expected distortion for various source-channel coding schemes

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