

Achievable Rates and Capacity for Gaussian Relay Channels with Correlated Noises

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Abstract—We investigate the Gaussian relay channel where the additive noises at the relay and destination are correlated. We obtain achievable rates for the compress-and-forward and decode-and-forward relaying strategies, and compare them to each other and to the capacity upper bound. We show that neither scheme is uniformly best over all channel gains and correlation coefficients. We also derive specific relationships between the channel gains and noise correlations for which one of these schemes is capacity-achieving, thereby increasing the class of relay channels for which capacity is known.

I. INTRODUCTION

It is well known that relays can help the transmission between a source and its destination to combat channel impairments such as multipath fading, shadowing, and path loss. To determine the capacity gain from relaying, the relay channel model was introduced in [1] and thoroughly studied in [2], where the two celebrated relaying strategies, decode-and-forward (DF) and compress-and-forward (CF), were introduced. A combination of both strategies was also discussed in [2], which yields so far the best coding strategy for the relay channel when simultaneous decoding is applied [3]. Furthermore, a max-flow min-cut upper bound on capacity was also derived in [2], which is still the best outer bound for the general three-node relay channel. However, the lower bounds given by the respective coding strategies discussed above do not meet the upper bound in general, and thus the relay channel capacity remains an open problem.

Although the capacity of the general relay channel is unknown, there are several special cases for which the capacity has been established: i) the physically degraded and the reversely degraded relay channels [2], ii) the semi-deterministic relay channel [4], iii) the phase fading relay channel [5], iv) the orthogonal relay channel [6], and v) a class of deterministic relay channels [7].

As one of the most important and practical cases of relay channels, the Gaussian relay channel has been considered in [2], [5], [6], [8], etc. All these works assume that the noises at the relay and the destination nodes are Gaussian and independent of each other. We consider a more general setting, where the relay and destination noises are arbitrarily correlated, as might happen when a common interference signal contributes to the noises at both receivers. We are

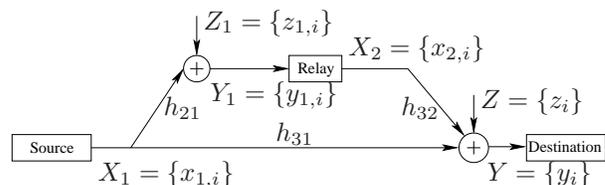


Fig. 1. Three-node Gaussian relay channel.

interested in finding which relaying strategy can best exploit the correlation. Since the DF strategy decodes the source message, it discards the correlated noise which might be of use to the destination, so this might hurt its performance. On the other hand CF removes certain information about the message via its compression, so this might also hurt its performance. Which strategy is best will depend on how much performance is lost due to the discarded information.

We will show in this work that neither relaying scheme is uniformly best over all channel gains and correlation coefficients. We also derive specific relationships between the channel gains and noise correlations where one of these relaying schemes is capacity-achieving.

The rest of the paper is organized as follows. In Section II we discuss the channel model. In Section III we review the existing coding strategies for the relay channel, derive the achievable rates with our channel configuration, and present the capacity theorems for two special classes of Gaussian relay channels with correlated noises. In Section IV we provide numerical examples to compare the various achievable rates in reference to the max-flow min-cut upper bound. We conclude the paper in Section V.

II. CHANNEL MODEL

Consider the Gaussian relay channel depicted in Fig. 1, where the source node transmits certain information to the destination node with the aid of the relay node. In a general setting, the channel inputs at the source node and relay node, $x_{t,i}$, $t = 1, 2$, respectively, are required to satisfy the individual power constraints, $\sum_{i=1}^n \|x_{t,i}\|^2/n \leq P_t$, $t = 1, 2$. As shown in Fig. 1, the channel gains denoted by h_{21} , h_{31} , and h_{32} , are assumed to be real scalars. We further assume that the relay experiences an additive white Gaussian noise, denoted

as $Z_1 \sim \mathcal{N}(0, N_1)$, and similarly for the destination, denoted as $Z \sim \mathcal{N}(0, N)$. We assume that noises at the relay and destination nodes are arbitrarily correlated with each other, with correlation coefficient

$$\rho_z := \frac{E\{Z_1 Z\}}{\sqrt{N_1 N}}.$$

The channel is fully described by the following two equations

$$\begin{aligned} y_{1,i} &= h_{21}x_{1,i} + z_{1,i}, \\ y_i &= h_{31}x_{1,i} + h_{32}x_{2,i} + z_i, \end{aligned}$$

at time instance i , where $y_{1,i}$ and y_i are the received signals at the relay and destination, respectively.

III. ACHIEVABLE RATES, CAPACITY UPPER BOUND, AND TWO NEW CAPACITY RESULTS

A. Existing Strategies for the Relay Channel

We will evaluate two well-known coding strategies for our channel model, the DF and the CF strategies. With DF, the relay node decodes the message from the source node, and encodes the message for retransmission. During the retransmission, the message is sent to the destination node through *cooperation*, which is enabled by the common information (the message to be retransmitted) at the source and relay nodes. The following achievable rate is established with the DF strategy [2]:

$$R_{\text{DF}} = \max_{p(x_1, x_2)} \min\{I(X_1; Y_1|X_2), I(X_1, X_2; Y)\}. \quad (1)$$

In the DF strategy, by completely decoding and re-encoding the message from the source node, the relay node discards the noise part contained in the source-relay channel output. We also observe from the two mutual information terms in (1) that the channel output at the relay node and the one at the destination node are treated separately. Hence, the correlation between the noises, if any, that is contained in the relay and destination channel outputs is completely ignored. This seems to imply that DF could be highly suboptimal under the correlated noise model. However, our later findings show that this is not true.

By contrast, the CF strategy does preserve the correlation between the channel noises to some extent. Specifically, in the CF strategy, the relay node does not decode any information sent from the source node. Instead, it compresses (quantizes) the source-relay channel output and forwards the compressed version to the destination node. This signal serves as the side information to the destination node when the destination decodes the source information from its own channel output. The following rate can be achieved with the CF strategy [2]:

$$R_{\text{CF}} = \sup_{p(\cdot) \in \mathcal{P}^*} I(X_1; Y, \hat{Y}_1|X_2), \quad (2)$$

subject to the constraint

$$I(X_2; Y) \geq I(Y_1; \hat{Y}_1|X_2, Y), \quad (3)$$

where \hat{Y} is an auxiliary random variable of a finite range, and \mathcal{P}^* denotes the set of distributions that factor in the form

$$p(x_1, x_2, y, y_1, \hat{y}_1) = p(x_1)p(x_2)p(y, y_1|x_1, x_2)p(\hat{y}_1|y_1, x_2).$$

In contrast to the DF strategy, the CF strategy performs a joint decoding over both the compressed channel output at the relay node and the one at the destination node, which can be observed from (2).

Next we give a simple example to demonstrate that, in some scenarios the CF strategy leads to a higher rate than the DF strategy due to its ability to preserve noise correlation.

Example 3.1: We assume fixed channel gains of 1, i.e., $h_{21} = h_{31} = h_{32} = 1$. Let the power limits at the source and relay nodes be $P_1 = 1$ and $P_2 = 36$, respectively. We also let the noises at the relay and destination nodes be perfectly correlated. In particular, we assume that $Z_1 + Z = 0$, and $Z_1, Z \sim \mathcal{N}(0, 5)$.

We first consider applying the DF strategy at the relay node. The achievable rate with DF is upper-bounded by the capacity of the source-relay link:

$$\begin{aligned} R_{\text{DF}} &\leq I(X_1; Y_1|X_2) \leq H(Y_1|X_2) - H(Y_1|X_2, X_1) \\ &= H(X_1 + Z_1|X_2) - H(Z_1) \leq h(X_1 + Z_1) - h(Z_1) \\ &= \frac{1}{2} \log_2 \left(1 + \frac{P_1}{N_1} \right) = \frac{1}{2} \log_2(1.2). \end{aligned}$$

Next we consider a simplified CF strategy. Specifically, in CF, the quantized signal is decoded from the channel outputs of both the present and previous blocks at the destination node, while in our simplified version, we decode the quantized signal from the channel output of the present block only. The relay node first quantizes the received signal with Wyner-Ziv coding [5], i.e., the source-relay channel output $Y_1 = X_1 + Z_1$ is quantized to $\hat{Y}_1 = X_1 + Z_1 + Z_{\text{WZ}}$ with the quantization noise Z_{WZ} being a Gaussian random variable of zero mean and unit variance, i.e., $Z_{\text{WZ}} \sim \mathcal{N}(0, 1)$. We now evaluate the mutual information terms as follows

$$I(\hat{Y}_1; Y_1|X_2) = \frac{1}{2} \log_2 \left(1 + \frac{P_1 + N_1}{N_{\text{WZ}}} \right) = \frac{1}{2} \log_2(7),$$

and

$$I(X_2; Y) = \frac{1}{2} \log_2 \left(1 + \frac{P_2}{P_1 + N} \right) = \frac{1}{2} \log_2(7).$$

Hence, we have $I(\hat{Y}_1; Y_1|X_2) \leq I(X_2; Y)$. Consequently, the quantized signal \hat{Y}_1 can be sent to the destination node reliably. After the destination successfully decodes the quantized information carried by X_2 , it subtracts X_2 from $Y = X_1 + X_2 + Z$. Thus the following signals are available:

$$\bar{Y} = X_1 + Z, \quad (4)$$

$$\hat{Y}_1 = X_1 + Z_1 + Z_{\text{WZ}}. \quad (5)$$

We observe from (4) and (5) that the correlated noises are well preserved in the refined signals \bar{Y} and \hat{Y}_1 at the destination. By summing (4) and (5), we obtain

$$Y_{\text{sum}} = 2X_1 + Z_{\text{WZ}}.$$

Hence, the following rate can be achieved

$$\tilde{R}_{\text{CF}} = \frac{1}{2} \log_2 \left(1 + \frac{4P_1}{N_{\text{WZ}}} \right) = \frac{1}{2} \log_2(5).$$

Note that since \tilde{R}_{CF} is just a simplified version of the CF achievable rate, we might get higher rates with CF by decoding the quantized signal from both the current and previous block. Thus, for this example, we have

$$R_{\text{DF}} = \frac{1}{2} \log_2(1.2) < \tilde{R}_{\text{CF}} = \frac{1}{2} \log_2(5) \leq R_{\text{CF}}. \quad (6)$$

Remark 3.1: As demonstrated in the above example, the CF strategy may offer certain advantages over the DF strategy in preserving and exploiting the correlation between the noises at the relay and destination nodes. However, the analysis in Section IV will show that the DF strategy may outperform the CF strategy in other system settings. Thus neither scheme has the best performance under all correlation models.

We next derive both the DF and CF achievable rates for the Gaussian relay channel with relay and destination noise correlation, as introduced in Section II.

B. Achievable Rates for the Gaussian Relay Channel With Correlated Relay and Destination Noises

Let us first define the following function $\Gamma(\alpha) := \frac{1}{2} \log_2(1 + \alpha)$. Define the correlation coefficient between the channel inputs at the source and relay nodes as

$$\rho_x := \frac{E\{X_1 X_2\}}{\sqrt{P_1 P_2}}.$$

First, we extend the DF rate derived for the discrete memoryless relay channel [2] to our channel setting as follows. Note that the achievable rate R'_{DF} is obtained by directly evaluating the DF rate in (1) with respect to the proposed channel model.

Proposition 3.1: For the Gaussian relay channel defined in Section II, the following rate is achievable with DF:

$$R'_{\text{DF}} := \max_{0 \leq \rho_x \leq 1} \min \left\{ \Gamma \left(\frac{h_{21}^2 P_1 (1 - \rho_x^2)}{N_1} \right), \Gamma \left(\frac{h_{31}^2 P_1 + h_{32}^2 P_2 + 2h_{31} h_{32} \rho_x \sqrt{P_1 P_2}}{N} \right) \right\}. \quad (7)$$

Remark 3.2: We observe that R'_{DF} is indeed not a function of the correlation coefficient ρ_x . As mentioned earlier, this is due to the fact that the relay node performs full-decoding of the source messages and re-encodes the decoded messages, which totally discards the correlated content in the relay and destination noises.

We next drive the CF rate with our channel configuration. Also note that the achievable rate R'_{CF} is obtained by directly evaluating the CF rate in (4) and (5).

Proposition 3.2: For the Gaussian relay channel defined in Section II, the following rate is achievable with CF:

$$R'_{\text{CF}} := \Gamma \left(\frac{P_1 h_{31}^2 N_1 + h_{31}^2 N_{\text{WZ}} + h_{21}^2 N - 2h_{21} h_{31} \rho_z \sqrt{N_1 N}}{(1 - \rho_z^2) N_1 + N_{\text{WZ}}} \right),$$

where N_{WZ} is computed as

$$\frac{h_{31}^2 P_1 N_1 + h_{21}^2 P_1 N + (1 - \rho_z^2) N_1 N - 2h_{21} h_{31} P_1 \rho_z \sqrt{N_1 N}}{h_{32}^2 P_2}.$$

Remark 3.3: In contrast to the DF rate in Proposition 3.1, the CF rate R'_{CF} is dependent on the correlation coefficient ρ_z between Z_1 and Z , the noises at the relay and destination nodes. As shown in Example 3.1, the compressed version of the source-relay channel output may help the destination node exploit the correlation between the noises in our channel model.

Remark 3.4: It has been noted in some existing work [5], [8] that if the source-relay link is weak, then direct transmissions from the source to the destination with no relaying can achieve an even larger rate than the DF rate. This is due to the fact that the DF strategy requires the relay node to fully decode the source message, which creates a bottleneck in the overall transmission. By contrast, the CF strategy does not have this problem. Note that the CF strategy always incorporates the direct transmission scheme as a special case: By setting \hat{Y}_1 as a constant, we have the constraint expressed in (3) always satisfied, and the CF achievable rate reduces to

$$I(X_1; Y, \hat{Y}_1 | X_2) = I(X_1; Y | X_2),$$

which is the capacity of the direct link between the source and destination nodes. The direct transmission rate with our channel setting can be easily computed as

$$R_{\text{Direct}} = I(X_1; Y | X_2) = \Gamma \left(\frac{h_{31}^2 P_1}{N} \right). \quad (8)$$

C. Max-Flow Min-Cut Upper Bound

The max-flow min-cut upper bound [2, Thm. 4] was derived for the general discrete memoryless relay channel, based on which we derive an upper bound for the proposed channel model as follows.

Proposition 3.3: For the Gaussian relay channel defined in Section II, an upper bound of the capacity is given by C^+ , where

$$C^+ := \max_{0 \leq \rho_x \leq 1} \min \left\{ \Gamma \left(\frac{h_{31}^2 P_1 + h_{32}^2 P_2 + 2h_{31} h_{32} \rho_x \sqrt{P_1 P_2}}{N} \right), \Gamma \left(\frac{(1 - \rho_x^2) P_1 (h_{21}^2 N + h_{31}^2 N_1 - 2h_{21} h_{31} \sqrt{N_1 N} \rho_z)}{(1 - \rho_z^2) N_1 N} \right) \right\}. \quad (9)$$

D. Capacity Results for Two Special Cases

In general, it is hard to find the capacity of the Gaussian relay channel with correlated relay and destination noises. However, we are able to do so for two special cases, where we have certain assumptions on the relationship between noise correlation and relative channel gains. Although the specific relationship is unlikely in practice, it does show that the relationship between noise correlation and relative channel gains impacts the capacity-achieving strategy.

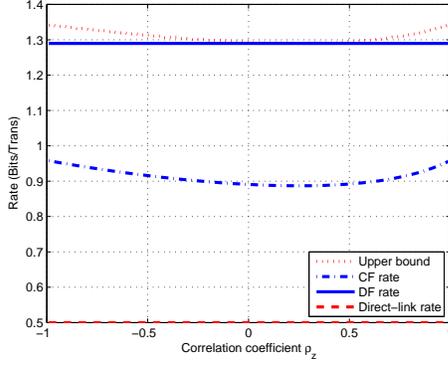


Fig. 2. Rates vs. the correlation coefficient ρ_z for $d = 0.25$.

Now let the correlated relay and destination noises be of the same variance, i.e., $N_1 = N = N_0$.¹

Theorem 3.1: For the relay channel defined in Section II with $N_1 = N = N_0$, when the correlation coefficient between the noises is equal to the ratio between the source-destination link and the source-relay link gains, i.e., $\rho_z = h_{31}/h_{21}$, the capacity of the channel can be achieved with the DF coding strategy. The capacity is given as:

$$C_1 = \max_{0 \leq \rho_x \leq 1} \min \left\{ \Gamma \left(\frac{h_{21}^2 P_1 (1 - \rho_x^2)}{N_0} \right), \Gamma \left(\frac{h_{31}^2 P_1 + h_{32}^2 P_2 + 2h_{31}h_{32}\rho_x \sqrt{P_1 P_2}}{N_0} \right) \right\}.$$

Proof: By substituting both N_1 and N with N_0 in (7) and (9), and substituting ρ_z with h_{31}/h_{21} in (9), we easily see that the DF rate R_{DF} becomes equal to the upper bound C^+ , and the theorem follows. ■

Theorem 3.2: For the relay channel defined in Section II with $N_1 = N = N_0$, when the correlation coefficient between the noises is equal to the ratio between the source-relay link and the source-destination link gains, i.e., $\rho_z = h_{21}/h_{31}$, the capacity of the channel can be achieved with direct source-destination transmissions (as a special case of CF). The capacity is given as

$$C_2 = \Gamma \left(\frac{h_{31}^2 P_1}{N_0} \right).$$

Proof: The achievability follows directly from (8), so we only need to show the converse.

The upper bound C^+ in (9) is further bounded by:

$$\begin{aligned} & \max_{0 \leq \rho_x \leq 1} \Gamma \left(\frac{(1 - \rho_x^2) P_1 (h_{21}^2 N + h_{31}^2 N_1 - 2h_{21}h_{31}\sqrt{N_1 N} \rho_x)}{(1 - \rho_x^2) N_1 N} \right) \\ & \leq \Gamma \left(\frac{P_1 (h_{21}^2 N + h_{31}^2 N_1 - 2h_{21}h_{31}\sqrt{N_1 N} \rho_x)}{(1 - \rho_x^2) N_1 N} \right). \end{aligned} \quad (10)$$

¹Note that a simple linear scaling can be applied to transform a channel defined in Section II to an equivalent channel with the relay and destination noises having the same variance, i.e., by multiplying the source-relay channel gain by $\sqrt{N/N_1}$ such that the scaled gain becomes $\sqrt{N/N_1}h_{21}$ and the noise variance at the relay becomes N .

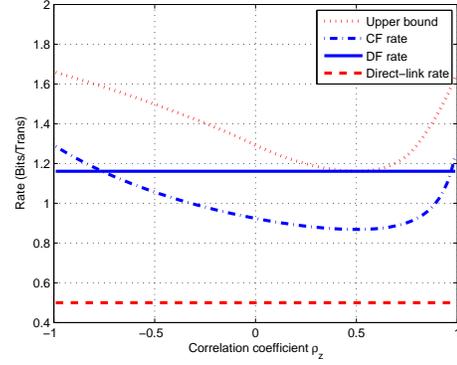


Fig. 3. Rates vs. the correlation coefficient ρ_z for $d = 0.50$.

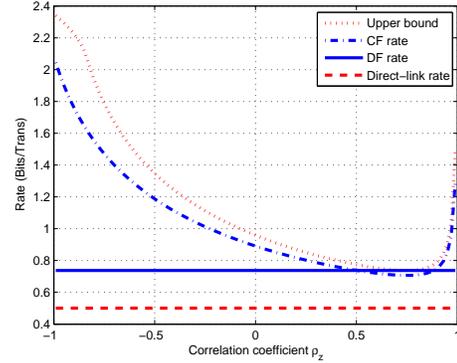


Fig. 4. Rates vs. the correlation coefficient ρ_z for $d = 0.75$.

In (10), by substituting both N_1 and N with N_0 , and substituting ρ_z with h_{21}/h_{31} , we have

$$C^+ \leq \Gamma \left(\frac{h_{31}^2 P_1}{N_0} \right),$$

which establishes the converse, and the theorem follows immediately. ■

IV. NUMERICAL ANALYSIS

We will evaluate the achievable rates under DF and CF as well as the max-flow min-cut upper bound for two channel configurations. These two configurations represent two different relationships between the channel gains, which will be specified later in this section. In addition, we set $P_1 = P_2 = N_1 = N = 1$ in the numerical examples for both configurations.

A. System Model I

In this model we assume that the channel amplitude gains are inversely proportional to the distance between the individual nodes, as in a free space path loss model. We set the distance from the source node to the destination node as 1, and assume that the relay node is located on the straight line connecting the source node and the destination node, i.e., the channel amplitude gains satisfy:

$$h_{21} = \frac{1}{d}, h_{32} = \frac{1}{1-d}, \text{ and } h_{31} = 1, \quad (11)$$

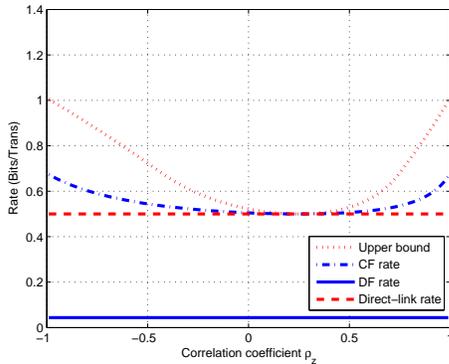


Fig. 5. Rates vs. the correlation coefficient ρ_z for $r = 0.25$.

where $d \in (0, 1)$ is the distance between the source and the relay nodes. With $d = 0.25$, $d = 0.50$, and $d = 0.75$, we plot the achievable rates including the DF rate, CF rate, and direct-link rate, together with the max-flow min-cut upper bound, versus the correlation coefficient ρ_z in Fig. 2, Fig. 3, and Fig. 4, respectively.

We observe from Figs. 2–4 that: 1) When the relay node is close to the source node, i.e., $d = 0.25$, DF outperforms CF over most of the correlation values, which is due to the fact that the signal-to-noise-ratio (SNR) at the relay is high such that it makes more sense to do DF rather than exploiting the noise correlation with CF; 2) when the relay node is close to the destination node, i.e. $d = 0.75$, the CF strategy outperforms DF and achieves rates relatively close to the upper bound over most of the correlation values. This is due to the fact that the SNR at the relay is low so that it becomes more beneficial to exploit the noise correlation with CF than with DF; 3) it appears in each figure that the DF rate meets the upper bound when we have

$$\rho_z = d = h_{31}/h_{21},$$

which verifies the capacity result claimed in Theorem 3.1.

B. System Model II

We note in the first configuration that the two channel links connecting to the relay are always better than the direct link from the source node to the destination node. This is the case particularly when the relay is placed between the source and destination nodes. However, if the relay node is far from both the other two nodes, the two links connecting to the relay are likely to be weaker than the direct link. Hence, we consider the following simplified channel setting:

$$h_{21} = r, h_{32} = 1 - r, \text{ and } h_{31} = 1, \quad (12)$$

where $r \in (0, 1)$. We plot the achievable rates and the upper bound versus the correlation coefficient between the noises for two different values of r in Figs. 5 and 6, respectively.

We observe from the two figures that the CF strategy always outperforms the DF strategy. This is mainly due to the fact that the source-relay link is always weaker than the direct link,

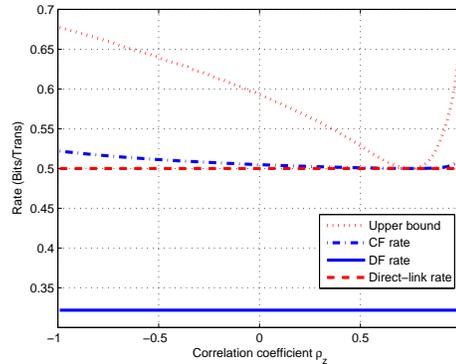


Fig. 6. Rates vs. the correlation coefficient ρ_z for $r = 0.75$.

which severely limits the DF rate. We also observe that for each channel realization defined by r , the CF rate and direct-link rate both meet the upper bound when we have

$$\rho_z = r = h_{21}/h_{31},$$

which verifies our claim in Theorem 3.2.

V. CONCLUSIONS

We have examined the capacity of the Gaussian relay channel when the relay and destination have correlated noises. This noise model introduces a new wrinkle in the analysis of decode-and-forward relaying, which removes the correlation information. We derive specific relationships between the channel gains and correlation coefficients for which decode-and-forward and compress-and-forward are capacity-achieving, thereby increasing the class of relay channels for which capacity is known. We also show the relative performance of these two strategies against each other and against the max-flow min-cut upper bound for a range of channel conditions and noise correlations. Our results indicate that the best relaying strategy varies depending on the channel and noise conditions.

REFERENCES

- [1] E. C. van der Meulen, "Three-terminal communication channels," *Adv. Appl. Prob.*, vol. 3, pp. 120–154, 1971.
- [2] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. 25, no. 5, pp. 572–584, Sept. 1979.
- [3] H. F. Chong, M. Motani, and H. K. Garg, "Generalized backward decoding strategies for the relay channel," *IEEE Trans. Inform. Theory*, vol. 53, no. 1, pp. 394–401, Jan. 2007.
- [4] A. El Gamal and M. Aref, "The capacity of the semideterministic relay channel," *IEEE Trans. Inform. Theory*, vol. IT-28, no. 3, p. 536, May 1982.
- [5] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inform. Theory*, vol. 51, no. 9, pp. 3037–3063, Sept. 2005.
- [6] A. El Gamal and S. Zahedi, "Capacity of a class of relay channels with orthogonal components," *IEEE Trans. Inform. Theory*, vol. 51, no. 5, pp. 1815–1817, May 2005.
- [7] T. M. Cover, "Capacity of a class of deterministic relay channels," in *Proc. IEEE Int. Symp. Information Theory*, Nice, France, June 24–29, 2007.
- [8] A. El Gamal, M. Mohseni, and S. Zahedi, "Bounds on capacity and minimum energy-per-bit for AWGN relay channels," *IEEE Trans. Inform. Theory*, vol. 52, no. 4, pp. 1545–1561, Apr. 2006.