# On Capacity Outer Bounds for a Simple Family of Wireless Networks

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***Abstract*—This paper explores models for finding outer bounds on the capacities of a simple family of wireless networks. Each wireless network is comprised of a collection of independent, memoryless channels with no more than three nodes: one transmitter and two receivers in each broadcast channel, two transmitters and one receiver in each multiple access channel, and one transmitter and one receiver in each point-to-point channel. The approach taken applies prior network equivalence results for modeling the individual components in each network and then bounds the difference between the modeling network capacity and the capacity of the original network. For binary channels, the modeling network guarantees capacities within a constant multiplicative factor of the true network capacity for all possible demand types. The results for networks of Gaussian channels yield cuts across each channel within an additive constant of the optimal cut value. These constant gaps in cut values give additive bounds on the accuracy of capacities for demand types where cut-sets are tight. The bounding network capacity is also tight for some example networks where the gap between the capacity region and the traditional cut-set outer bounds can be made arbitrarily large.**

## I. Introduction

Cut-set bounds [1, Theorem 15.10.1] are a powerful tool for bounding capacities of network systems. Unfortunately, these bounds are known not to be tight even for some very simple network examples [1, Section 15.10], and the gap between the cut-set bounds and capacity can grow linearly with the network size even for networks of noiseless, point-to-point links [2]. Combining recent work on network equivalence [3], [4] with old and new analytical (e.g., [5], [6], [2]) and computational (e.g., [7], [8]) bounds on capacities of networks of noiseless, point-to-point channels yields many new upper bounds on the capacities of networks. A corresponding lower bound on each network capacity can be achieved by first channel coding across each individual channel and then applying available lower bounds on the capacities of the resulting network of (asymptotically) noiseless point-to-point links.[1] Combining these approaches yields network capacity bounds and corresponding bounds on their accuracy.

This paper explores this approach on two families of wireless network models: a family of binary networks and a family of Gaussian networks. Each family includes all possible networks that can be built from a small collection of independent, memoryless point-to-point, two-receiver broadcast, and additive two-transmitter multiple access channels with additive noise. Networks with and without cycles are included in this class. This simple family of network models incorporates both the broadcast and interference features of wireless communication networks. Bounds on the capacity of each network are given as a function of the network coding capacity of a corresponding network of noiseless, point-to-point links. These results are tight in some cases – including networks where separation is known not to be optimal. For the binary channel models, all capacities for the bounding network are shown to be within a constant multiplicative factor of the true network capacity; this result holds even for demand types like multiple unicast demands where precise characterization of capacity remains an open problem. For the Gaussian channel models, additive constant bounds on the values of cuts yield additive constant bounds on network capacity for demand types where cut-sets are tight.

Section II gives the problem set-up and background on network models. Section III describes the channels employed here and derives their upper and lower bounding models. Section IV bounds the accuracy of the modeling network capacities as estimates for the true network capacity.

## II. Preliminaries

Consider an $m$-node network $\mathcal{N}$. At each time $t$, node $v$ transmits a random variable $X_t^{(v)}$ and receives a random variable $Y_t^{(v)}$. The network is memoryless, so it is characterized by a conditional probability distribution

$$p(\mathbf{y}|\mathbf{x}) = p(y^{(1)}, \ldots, y^{(m)}|x^{(1)}, \ldots, x^{(m)}).$$

The network structure is described by a hypergraph with node set $V = \{1, \ldots, m\}$ and hyperedge set $E = \{e_1, \ldots, e_k\}$. Each hyperedge $e = [V_1, V_2]$ has a collection of input nodes $V_1 \subset V$ and a set of output nodes $V_2 \subseteq V \setminus V_1$. This work focuses on networks of point-to-point ($|V_1| = |V_2| = 1$), two-receiver broadcast channels ($|V_1| = 1$, $|V_2| = 2$), and two-transmitter multiple access channels ($|V_1| = 2$, $|V_2| = 1$). The indegree and outdegree of node $v$ are

$$\begin{aligned} d_{\text{in}}(v) &\stackrel{\text{def}}{=} |\{[V_1, V_2] \in E | v \in V_2\}| \\ d_{\text{out}}(v) &\stackrel{\text{def}}{=} |\{[V_1, V_2] \in E | v \in V_1\}|. \end{aligned}$$

[1]This observation is almost immediate for networks without cycles. The argument for networks with cycles requires more care because we cannot apply block channel codes across time in these examples. See [3] and [4] for details in the cases of point-to-point channel components and network channel components, respectively.

When $d_{\text{out}}(v)$ and $d_{\text{in}}(v)$ exceed 1, $\mathcal{X}^{(v)} = \prod_{d=1}^{d_{\text{out}}(v)} \mathcal{X}^{(v,d)}$, $\mathcal{Y}^{(v)} = \prod_{d=1}^{d_{\text{in}}(v)} \mathcal{Y}^{(v,d)}$, $X_t^{(v)} = (X_t^{(v,1)}, \ldots, X_t^{(v,d_{\text{out}}(v))})$, and $Y_t^{(v)} = (Y_t^{(v,1)}, \ldots, Y_t^{(v,d_{\text{in}}(i))})$, where the outputs and inputs of $v$ are described in the (fixed but arbitrary) order imposed by the hyperedge indices. Let $V_1(e)$ and $V_2(e)$ denote the input and output ports of edge $e$. For example, if channel $e = [\{u\}, \{v_1, v_2\}]$ has input port $(u, s)$ for some $s \in \{1, \ldots, d_{\text{out}}(u)\}$ and output ports $(v_1, r_1)$ and $(v_2, r_2)$ for some $r_i \in \{1, \ldots, d_{\text{in}}(v_i)\}$, $i \in \{1, 2\}$, then $V_1(e) = \{(u, s)\}$ and $V_2(e) = \{(v_1, r_1), (v_2, r_2)\}$. For any set $S \subseteq \{(u, s) : u \in \{1, \ldots, m\}, s \in \{1, \ldots, d_{\text{out}}(u)\}$, let $X^S \stackrel{\text{def}}{=} (X^{(v)} : v \in S)$ denote the vector of channel inputs described by $S$, and let $x^S$ specify a single instance of that random vector. Similarly, for any set $S \subseteq \{(u, s) : u \in \{1, \ldots, m\}, s \in \{1, \ldots, d_{\text{in}}(u)\}$, $Y^S \stackrel{\text{def}}{=} (Y^{(v)} : v \in S)$ denotes a vector of channel outputs, and $y^S$ is a particular instance of that vector. When $S$ is empty, $X^S$ and $Y^S$ are treated as fixed but arbitrary constants. The channels described by the elements of $E$ are independent, so

$$p(\mathbf{y}|\mathbf{x}) = \prod_{e \in E} p(y^{V_2(e)} | x^{V_1(e)}).$$

A blocklength-$n$ code operates the network over $n$ time steps with the goal of communicating, for all distinct $(v, v')$, message $W^{(v \to v')} \in \mathcal{W}^{(v \to v')} \stackrel{\text{def}}{=} \{1, \ldots, 2^{nR^{(v \to v')}}\}$ from node $v$ to node $v'$. For any distinct pairs of vertices, $(u, u'), (v, v') \in V^2$, the messages $W^{(u \to u')}$ and $W^{(v \to v')}$ may be identical (for example in a multicast problem with $v = u$ and $v' \neq u'$) or independent. The vector of rates $R^{(v \to v')}$ is denoted by $\mathcal{R}$. A network is thus written as a triple

$$\left( \prod_{v=1}^{m} \mathcal{X}^{(m)}, \prod_{e \in E} p(y^{V_2(e)} | x^{V_1(e)}), \prod_{v=1}^{m} \mathcal{Y}^{(v)} \right)$$

with the added constraint that $X_t^{(v)}$ is a function of $\{Y_1^{(v)}, \ldots, Y_{t-1}^{(v)}, W^{(v \to 1)}, \ldots, W^{(v \to m)}\}$ alone.

*Remark 1:* Including a "no transmission" symbol in the input and output alphabets of each channel allows arbitrary scheduling within this time-step model.

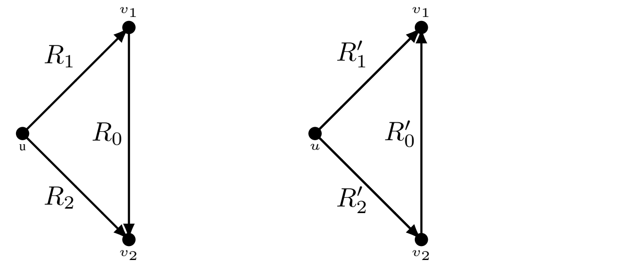
*Definition 1:* Let a network

$$\mathcal{N} \stackrel{\text{def}}{=} \left( \prod_{v=1}^{m} \mathcal{X}^{(v)}, \prod_{e \in E} p(y^{V_2(e)} | x^{V_1(e)}), \prod_{v=1}^{m} \mathcal{Y}^{(v)} \right)$$

be given corresponding to a graph $G = (V, E)$. A blocklength-$n$ solution $\mathcal{S}(\mathcal{N})$ to this network is a set of encoding and decoding functions:

$$X_t^{(v)} : (\mathcal{Y}^{(v)})^{t-1} \times \prod_{v'=1}^{m} \mathcal{W}^{(v \to v')} \to \mathcal{X}^{(v)}$$
$$\hat{W}^{(v' \to v)} : (\mathcal{Y}^{(v)})^{n} \times \prod_{v'=1}^{m} \mathcal{W}^{(v \to v')} \to \mathcal{W}^{(v' \to v)}$$

mapping $(Y_1^{(v)}, \ldots, Y_{t-1}^{(v)}, W^{(v \to 1)}, \ldots, W^{(v \to m)})$ to $X_t^{(v)}$ and $(Y_1^{(v)}, \ldots, Y_n^{(v)}, W^{(v \to 1)}, \ldots, W^{(v \to m)})$ to $\hat{W}^{(v' \to v)}$. Solution $\mathcal{S}(\mathcal{N})$ is called a $(\lambda, \mathcal{R})$-solution, denoted $(\lambda, \mathcal{R}) - \mathcal{S}(\mathcal{N})$, if the encoding and decoding functions imply $\Pr(W^{(v \to v')} \neq \hat{W}^{(v \to v')}) < \lambda$ for all $v, v'$.




$R_0 = I(X^{(u,s)}; Y^{(v_1,r_1)})$ $\qquad R'_0 = I(X^{(u,s)}; Y^{(v_2,r_2)})$
$R_1 = I(X^{(u,s)}; Y^{(v_1,r_1)})$ $\qquad R'_1 = I(X^{(u,s)}; Y^{(v_1,r_1)} | Y^{(v_2,r_2)})$
$R_2 = I(X^{(u,s)}; Y^{(v_2,r_2)} | Y^{(v_1,r_1)})$ $\qquad R'_2 = I(X^{(u,s)}; Y^{(v_2,r_2)})$

Fig. 1. Two deterministic broadcast channel models from [4].

*Definition 2:* The rate region $\mathscr{R}(\mathcal{N}) \subset \mathbb{R}_+^{m(m-1)}$ of a network $\mathcal{N}$ is the closure of the set of rate vectors $\mathcal{R}$ such that for any $\lambda > 0$ a $(\lambda, \mathcal{R}) - \mathcal{S}(\mathcal{N})$ solution exists for all blocklengths $n$ sufficiently large.

Theorem 2.1 bounds the capacity of a network $\mathcal{N}$ by the capacity of a network $\mathcal{N}_U$ that is identical to $\mathcal{N}$ except that a single noisy channel in $\mathcal{N}$ is replaced by a noiseless bit-pipe of the same capacity.

*Theorem 2.1 ([3, Theorem 3]):* Let networks $\mathcal{N}$ and $\mathcal{N}_U$ be defined as

$$\begin{aligned} \mathcal{N} \;=\; & \Big( \mathcal{X}^{(1,1)} \times \cdots \times \mathcal{X}^{(u,s)} \times \cdots \times \mathcal{X}^{(m,d_{\text{out}}(m))}, \\ & p(y^{(v,r)} | x^{(u,s)}) \prod_{e \in E \setminus \{\bar{e}\}} p(y^{V_2(e)} | x^{V_1(e)}), \\ & \mathcal{Y}^{(1,1)} \times \cdots \times \mathcal{Y}^{(v,r)} \times \cdots \times \mathcal{Y}^{(m,d_{\text{in}}(m))} \Big) \end{aligned}$$

$$\begin{aligned} \mathcal{N}_U \;=\; & \Big( \mathcal{X}^{(1,1)} \times \cdots \times \hat{\mathcal{X}}^{(u,s)} \times \cdots \times \mathcal{X}^{(m,d_{\text{out}}(m))}, \\ & \delta(\hat{x}^{(u,s)} - \hat{y}^{(v,r)}) \prod_{e \in E \setminus \{\bar{e}\}} p(y^{V_2(e)} | x^{V_1(e)}), \\ & \mathcal{Y}^{(1,1)} \times \cdots \times \hat{\mathcal{Y}}^{(v,r)} \times \cdots \times \mathcal{Y}^{(m,d_{\text{in}}(m))} \Big), \end{aligned}$$

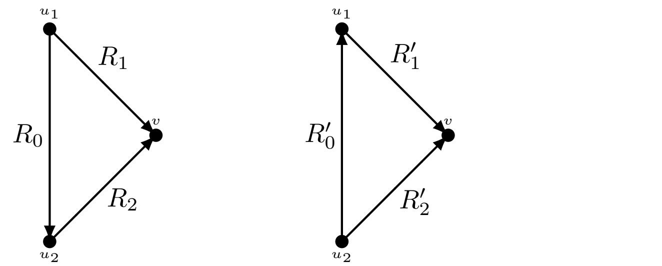
where $(\hat{\mathcal{X}}^{(u,s)}, \delta(\hat{x}^{(u,s)} - \hat{y}^{(v,r)}), \hat{\mathcal{Y}}^{(v,r)})$ is a bit pipe that noiselessly maps $\max_{p(x^{u,s})} I(X^{(u,s)}; Y^{(v,r)})$ bits from its input to its output at each time step. Then

$$\mathscr{R}(\mathcal{N}) = \mathscr{R}(\mathcal{N}_U).$$

Theorem 2.2 bounds the capacity of a network $\mathcal{N}$ by the capacity of a network $\mathcal{N}_U$ that is identical to $\mathcal{N}$ except for the replacement a single broadcast channel from node $u$ to nodes $v_1$ and $v_2$ by a collection of noiseless, capacitated bit-pipes configured as shown in Figure 1.

*Theorem 2.2 ([4, Theorem 4]):* Given a network

$$\begin{aligned} \mathcal{N} \;=\; & \Big( \mathcal{X}^{(1,1)} \times \cdots \times \mathcal{X}^{(u,s)} \times \cdots \times \mathcal{X}^{(m,d_{\text{out}}(m))}, \\ & p(y^{(v_1,r_1)}, y^{(v_2,r_2)} | x^{(u,s)}) \prod_{e \in E \setminus \{\bar{e}\}} p(y^{V_2(e)} | x^{V_1(e)}), \\ & \mathcal{Y}^{(1,1)} \times \cdots \times \mathcal{Y}^{(v_1,r_1)} \times \cdots \times \mathcal{Y}^{(v_2,r_2)} \times \cdots \\ & \times \mathcal{Y}^{(m,d_{\text{in}}(m))} \Big), \end{aligned}$$




$R_0 = \log|\mathcal{X}^{(u_1,s_1)}|$ $\qquad$ $R'_0 = \log|\mathcal{X}^{(u_2,s_2)}|$
$R_1 = I(X^{(u_1,s_1)};U)$ $\qquad$ $R'_1 = I(X^{(u_1,s_1)}, X^{(u_2,s_2)}; Y^{(v,r)}|U)$
$R_2 = I(X^{(u_1,s_1)}, X^{(u_2,s_2)}; Y^{(v,r)}|U)$ $\qquad$ $R_2 = I(X^{(u_2,s_2)};U)$

Fig. 2. Two deterministic multiple access channel models from [4].

let $d_u = d_{\text{out}}(u)+1$, $d_1 = d_{\text{out}}(v_1)+1$, and $d_2 = d_{\text{in}}(v_2)+1$. For each marginal $p = (p(x^{(u,s)}))_{x^{(u,s)}}$, define network $\mathcal{N}_U(p)$

$$\begin{aligned}\mathcal{N}_U(p) \;=\; & \Big(\mathcal{X}^{(1,1)} \times \cdots \times \hat{\mathcal{X}}^{(u,s)} \times \cdots \times \hat{\mathcal{X}}^{(u,d_u)} \times \cdots \\ & \times \hat{\mathcal{X}}^{(v_1,d_1)} \times \cdots \times \mathcal{X}^{(m,d_{\text{out}}(m))}, \\ & \delta(\hat{x}^{(u,s)} - \hat{y}^{(v_1,r_1)})\delta(\hat{x}^{(u,d_u)} - \hat{y}^{(v_2,r_2)}) \\ & \cdot\delta(\hat{x}^{(v_1,d_1)} - \hat{y}^{(v_2,d_2)}) \prod_{e\in E\setminus\{\bar{e}\}} p(y^{V_2(e)}|x^{V_1(e)}), \\ & \mathcal{Y}^{(1,1)} \times \cdots \times \hat{\mathcal{Y}}^{(v_1,r_1)} \times \cdots \times \hat{\mathcal{Y}}^{(v_2,r_2)} \times \cdots \\ & \times \hat{\mathcal{Y}}^{(v_2,d_2)} \times \cdots \times \mathcal{Y}^{(m,\hat{d}_{\text{in}}(m))}\Big),\end{aligned}$$

where

$$\begin{aligned} & (\hat{\mathcal{X}}^{(v_1,d_1)}, \delta(\hat{x}^{(v_1,d_1)} - \hat{y}^{(v_2,d_2)}), \hat{\mathcal{Y}}^{(v_2,d_w)}) \\ & (\hat{\mathcal{X}}^{(u,s)}, \delta(\hat{x}^{(u,s)} - \hat{y}^{(v_1,r_1)}), \hat{\mathcal{Y}}^{(v_1,r_1)}) \\ & (\hat{\mathcal{X}}^{(u,d_u)}, \delta(\hat{x}^{(u,d_u)} - \hat{y}^{(v_2,r_2)}), \hat{\mathcal{Y}}^{(v_2,r_2)})\end{aligned}$$

are, respectively, noiseless bit pipes of capacities

$$\begin{aligned} R_0 &= I(X^{(u,s)}; Y^{(v_1,r_1)}) \\ R_1 &= I(X^{(u,s)}; Y^{(v_1,r_1)}) \\ R_2 &= I(X^{(u,s)}; Y^{(v_2,r_2)}|Y^{(v_1,r_1)}),\end{aligned}$$

for distribution $p(x^{(u,s)})p(y^{(v_1,r_1)}, y^{(v_2,r_2)}|x^{(u,s)})$. Then

$$\mathscr{R}(\mathcal{N}) \subseteq \bigcup_p \mathscr{R}(\mathcal{N}_U(p)).$$

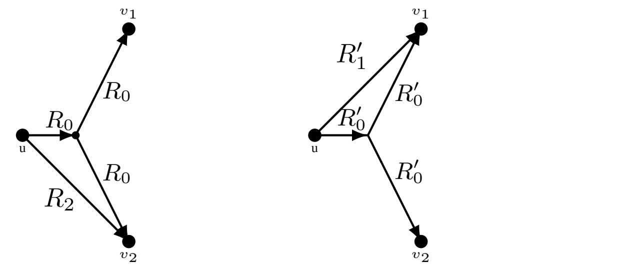
Theorem 2.3 bounds the capacity of a network $\mathcal{N}$ by the capacity of a network $\mathcal{N}_U$ that is identical to $\mathcal{N}$ except that a single multiple access channel from nodes $u_1$ and $u_2$ to node $v$ is replaced by a collection of noiseless, capacitated bit pipes as shown in Figure 2.

*Theorem 2.3 ([4, Theorem 3]):* Given a network

$$\begin{aligned}\mathcal{N} \;=\; & \Big(\mathcal{X}^{(1,1)} \times \cdots \times \mathcal{X}^{(u_1,s_1)} \times \cdots \times \mathcal{X}^{(u_2,s_2)} \times \\ & \cdots \times \mathcal{X}^{(m,d_{\text{out}}(m))}, \\ & p(y^{(v,r)}|x^{(u_1,s_1)}, x^{(u_2,s_2)}) \prod_{e\in E\setminus\{\bar{e}\}} p(y^{V_2(e)}|x^{V_1(e)}), \\ & \mathcal{Y}^{(1,1)} \times \cdots \times \mathcal{Y}^{(v,r)} \times \cdots \times \mathcal{Y}^{(m,d_{\text{in}}(m))}\Big),\end{aligned}$$

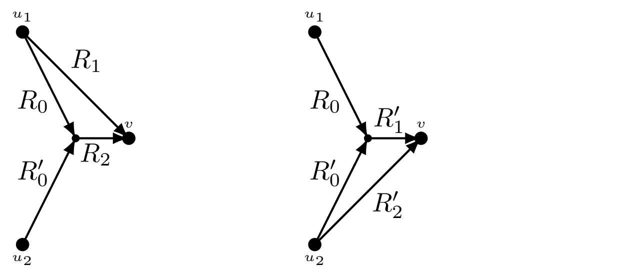
let $d_1 = d_{\text{out}}(u_1)+1$, $d_2 = d_{\text{in}}(u_2)+1$, and $d_v = d_{\text{in}}(v)+1$. For each distribution

$$p = (p(x^{(u_1,s_1)}, x^{(u_2,s_2)}))_{x^{(u_1,s_1)}, x^{(u_2,s_2)}},$$




$R_0 = I(X^{(u,s)};Y^{(v_1,r_1)})$ $\qquad$ $R'_0 = I(X^{(u,s)};Y^{(v_2,r_2)})$
$R'_1 = I(X^{(u,s)};Y^{(v_1,r_1)}|Y^{(v_2,r_2)})$
$R_2 = I(X^{(u,s)};Y^{(v_2,r_2)}|Y^{(v_1,r_1)})$

Fig. 3. Alternative deterministic broadcast channel models.




$R_0 = \log|\mathcal{X}^{(u_1,s_1)}|$ $\qquad$ $R'_0 = \log|\mathcal{X}^{(u_2,s_2)}|$
$R_1 = I(X^{(u_1,s_1)};U)$ $\qquad$ $R'_1 = I(X^{(u_1,s_1)}, X^{(u_2,s_2)}; Y^{(v,r)}|U)$
$R_2 = I(X^{(u_1,s_1)}, X^{(u_2,s_2)}; Y^{(v,r)}|U)$ $\qquad$ $R_2 = I(X^{(u_2,s_2)};U)$

Fig. 4. Alternative deterministic multiple access channel models.

define network $\mathcal{N}_U(p)$

$$\begin{aligned}\mathcal{N}_U(p) \;=\; & \Big(\mathcal{X}^{(1,1)} \times \cdots \times \hat{\mathcal{X}}^{(u_1,s_1)} \times \cdots \times \hat{\mathcal{X}}^{(u_1,d_1)} \times \cdots \\ & \times \hat{\mathcal{X}}^{(u_2,s_2)} \times \cdots \times \mathcal{X}^{(m,d_{\text{out}}(m))}, \\ & \delta(\hat{x}^{(u_1,s_1)} - \hat{y}^{(v,r)})\delta(\hat{x}^{(u_1,d_1)} - \hat{y}^{(u_2,d_2)}) \\ & \cdot\delta(\hat{x}^{(u_2,s_2)} - \hat{y}^{(v,d_v)}) \prod_{e\in E\setminus\{\bar{e}\}} p(y^{V_2(e)}|x^{V_1(e)}), \\ & \mathcal{Y}^{(1,1)} \times \cdots \times \hat{\mathcal{Y}}^{(u_2,d_2)} \times \cdots \times \hat{\mathcal{Y}}^{(v,r)} \times \cdots \\ & \times \hat{\mathcal{Y}}^{(v,d_v)} \times \cdots \times \mathcal{Y}^{(m,\hat{d}_{\text{in}}(m))}\Big),\end{aligned}$$

where

$$\begin{aligned} & (\hat{\mathcal{X}}^{(v_1,d_1)}, \delta(\hat{x}^{(v_1,d_1)} - \hat{y}^{(v_2,d_2)}), \hat{\mathcal{Y}}^{(v_2,d_2)}) \\ & (\hat{\mathcal{X}}^{(u_1,s_1)}, \delta(\hat{x}^{(u_1,s_1)} - \hat{y}^{(v,r)}), \hat{\mathcal{Y}}^{(v,r)}) \\ & (\hat{\mathcal{X}}^{(u_2,d_v)}, \delta(\hat{x}^{(u_2,d_2)} - \hat{y}^{(v,d_v)}), \hat{\mathcal{Y}}^{(v,d_v)})\end{aligned}$$

are, respectively noiseless bit pipes of capacities

$$\begin{aligned} R_0 &= \log|\mathcal{X}^{(u_1,s_1)}| \\ R_1 &= I(X^{(u_1,s_1)}; U) \\ R_2 &= I(X^{(u_1,s_1)}, X^{(u_2,s_2)}; Y^{(v,r)}|U),\end{aligned}$$

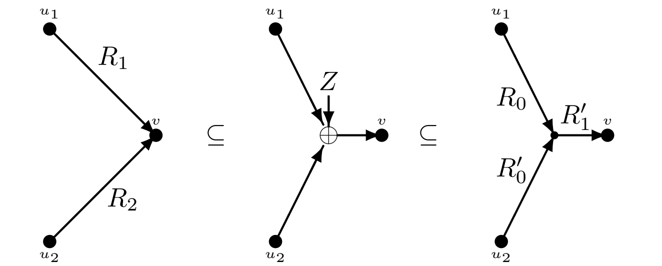
for distribution

$$p(x^{(u_1,s_1)}, x^{(u_2,s_2)})p(u|x^{(u_1,s_1)})p(y^{(v,r)}|x^{(u_1,s_1)}, x^{(u_2,s_2)}).$$

Here $\log|\mathcal{X}^{(u_1,s_1)}| = \infty$ if $\mathcal{X}^{(u_1,s_1)}$ is continuous. Then

$$\mathscr{R}(\mathcal{N}) \subseteq \bigcup_p \mathscr{R}(\mathcal{N}_U(p)).$$

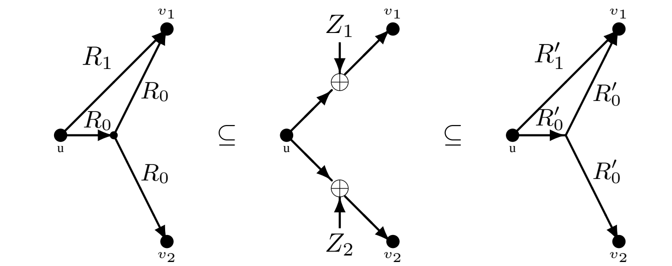
The multiple access and broadcast models are asymmetrical in their broadcast receivers or multiple access transmitters.




$R_1 = \alpha(1 - H(p))$
$R_2 = (1 - \alpha)(1 - H(p))$

$R_0 = R'_0 = 1$
$R'_1 = 1 - H(p)$

Fig. 5. Example upper and lower bounding models for the noisy binary adder multiple access channel with error probability $p$. The given models are lower and upper bounds in the sense that for any network $\mathcal{N}$ containing the given multiple access channel $\bar{e}$, replacing $\bar{e}$ by the model shown on its left yields a new network $\mathcal{N}_{L,\alpha}$ whose capacity region is a subset of the capacity region of the original network ($\mathscr{R}(\mathcal{N}_{L,\alpha}) \subseteq \mathscr{R}(\mathcal{N})$). Likewise, replacing $\bar{e}$ by the model shown on its right yields a new network $\mathcal{N}_U$ whose capacity region is a superset of the capacity region of the original network ($\mathscr{R}(\mathcal{N}) \subseteq \mathscr{R}(\mathcal{N}_U)$).




$R_0 = 1 - H(p_1 * p_1 * p_2 * p_2)$ $\quad R'_0 = 1 - H(p_1 * p_2)$
$R_1 = H(p_1 * p_1 * p_2) - H(p_1)$ $\quad R'_1 = H(p_1 * p_1 * p_2) - H(p_1)$
(a)

$R_0 = 1 - H(p_1 * p_2 * p_2)$ $\quad R'_0 = 1 - H(p_1 * p_2)$
$R_1 = H(p_1 * p_2) - H(p_1)$ $\quad R'_1 = H(p_1 * p_2) - H(p_1)$
(b)

Fig. 6. Example upper and lower bounding models for the binary symmetric broadcast channel with error probabilities $p_1$ and $p_1 * p_2$ at its two receivers. The link capacities given in (a) and (b) correspond to the independent noise and physically degraded cases, respectively.

Since the labeling is arbitrary, these results effectively offer two models for each broadcast channel and two models for each multiple access channel. Both models for each component guarantee outer bounds on the capacity region of the original network, and therefore choosing the model yielding the smallest capacity gives the tightest bound. The best model choice may vary with the demand type.

In order to avoid creating artificial paths between the receivers of broadcast channels and the transmitters of multiple access channels, the discussion that follows replaces the models in Figures 1 and 2 by those in Figures 3 and 4. Since the alternative models can deliver precisely the same rate from each transmitter to each receiver, the same proofs apply.

## III. Network Models

This section describes inner and outer bounding models for a simple collection of memoryless additive noise channels. For the first three channel types, the input and output alphabets are binary. For the remaining channel types, the input and output alphabets are continuous. The inner bounds correspond to points on the channel capacity regions. The derived outer bounds follow from the results in Section II.

### A. Binary Symmetric Channel

Let $\mathcal{E}_1^w$ be the class of binary symmetric point-to-point channels. Then any $\bar{e} \in \mathcal{E}_1^w$ describes a channel $(\{0,1\}, p(y^{(v,r)}|u^{(u,s)}), \{0,1\})$ with $Y^{(v,r)} = X^{(u,s)} \oplus Z$ and $EZ = p$. As is the case for all point-to-point channels [3], the inner and outer bounding models for this channel match precisely. In this case, each is a capacitated noiseless bit-pipe of capacity $R = 1 - H(p)$.

### B. Binary Adder Multiple Access Channel

Let $\mathcal{E}_1^m$ be the class of binary adder multiple access channels. Then any hyperedge $\bar{e} \in \mathcal{E}_1^m$ describes a channel

$$(\{0,1\}^2, p(y^{(v,r)}|x^{(u_1,s_1)}, x^{(u_2,s_2)}), \{0,1\})$$

with $Y^{(v,r)} = X^{(u_1,s_1)} \oplus X^{(u_2,s_2)} \oplus Z$. Let $EZ = p$. Figure 5 shows lower and upper bounding models for $\bar{e}$. Each lower bounding model comes from a point on the capacity region. The upper bound evaluates the model in Figure 4 with $U = c$.

The models for this example are quite intuitive. For any network $\mathcal{N}$ containing the binary adder multiple access channel $\bar{e}$, let $\mathcal{N}_{L,\alpha}$ and $\mathcal{N}_U$ designate the networks that result when the channel $\bar{e}$ is replaced by the lower bounding model with parameter $\alpha \in [0,1]$ and the upper bounding model, respectively. Applying a channel code across the multiple access channel in $\mathcal{N}$ allows us to transmit $(\alpha(1-H(p)), (1-\alpha)(1-H(p)))$ bits per channel use across channel $\bar{e}$ with an error probability that can be made arbitrarily small. Thus any code designed for network $\mathcal{N}_{L,\alpha}$ can be operated across $\mathcal{N}$ with an asymptotically small difference in error probability. As a result, $\mathscr{R}(\mathcal{N}_{L,\alpha}) \subseteq \mathscr{R}(\mathcal{N})$ for all $\alpha$, giving $\cup_\alpha \mathscr{R}(\mathcal{N}_{L,\alpha}) \subseteq \mathscr{R}(\mathcal{N})$. Similarly, any code designed for network $\mathcal{N}$ can be operated on network $\mathcal{N}_U$ by implementing a memoryless binary adder at the central node. Thus $\mathscr{R}(\mathcal{N}) \subseteq \mathscr{R}(\mathcal{N}_U)$.

### C. Binary Symmetric Broadcast Channel

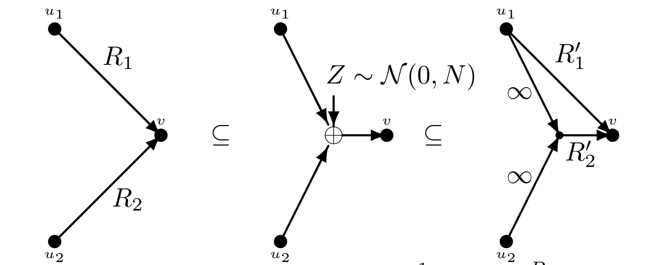
Let $\mathcal{E}_1^b$ be the class of binary symmetric broadcast channels Then any hyperedge $\bar{e} \in \mathcal{E}_1^b$ describes a channel

$$(\{0,1\}, p(y^{(v_1,r_1)}, y^{(v_2,r_2)}|x^{(u,s)}), \{0,1\}^2)$$

with $Y^{(v_1,r_1)} = X^{(u,s)} \oplus Z_1$ and $Y^{(v_2,r_2)} = X^{(u,s)} \oplus Z_2$ as shown in Figure 6. Let $p_1 = EZ_1$ and $p_1 * p_2 = p_1(1-p_2) + p_2(1-p_1) = EZ_2$. Figure 6 shows some example bounding networks when $Z_1$ and $Z_2$ are independent (a) and when the channel is physically degraded (b).

The two lower bounding networks come from the broadcast capacity, which delivers common rate $R_0 = I(U; Y^{(s_2,r_2)}) = 1 - H(\alpha * p_1 * p_2)$ to both receivers and individual rate $R_1 = I(X^{(u,s)}; Y^{(v_1,r_1)}|U) = H(a * p_1) - H(p_1)$ to receiver 1. This provides a whole family of possible lower bounds. If $\mathcal{N}_{L,\alpha}$ designates the lower bound using parameter $\alpha$, then $\cup_\alpha \mathscr{R}(\mathcal{N}_{L,\alpha}) \subseteq \mathscr{R}(\mathcal{N})$. For simplicity, (a) and (b) show the models corresponding to $\alpha = p_1 * p_2$ and $\alpha = p_2$, respectively.

Each upper bounding network is obtained by evaluating the model shown in Figure 3 under the given assumptions on $Z_1$




$R_1 = \frac{1}{2}\log(1 + \frac{P_1}{N})$ $\quad$ $R_1' = \frac{1}{2}\log(1 + \frac{P_1}{N})$

$R_2 = \frac{1}{2}\log(1 + \frac{P_2}{P_1+N})$ $\quad$ $R_2' = \frac{1}{2}\log(1 + \frac{2\sqrt{P_1P_2}+P_2}{N+P_1+P_2})$

Fig. 7. Example models for the Gaussian multiple access channel with power constraints $P_1 \geq P_2$ at transmitters 1 and 2 and variance-$N$ Gaussian noise.

and $Z_2$. The upper bounding model in (a) is larger than that in (b) because when $Z_1$ and $Z_2$ are independent, $Y^{(v_1,r_1)}$ and $Y^{(v_2,r_2)}$ together provide more information about $X^{(u,s)}$ than $Y^{(v_1,r_1)}$ alone; this is not true when the channel is physically degraded. Since the network in which the channel is employed may later take advantage of this additional information, any generic outer bound for the broadcast channel with independent noise must capture the potential for this channel to carry more than the maximal sum-rate of its capacity region.

### D. Gaussian Channel

Let $\mathcal{E}_2^w$ be the class of Gaussian noise channels. Then edge $\bar{e} \in \mathcal{E}_2^w$ is a channel $(\mathbb{R}, p(y^{(v,r)}|u^{(u,s)}), \mathbb{R})$ where $Y^{(v,r)} = X^{(u,s)} + Z$. Let $Z \sim \mathcal{N}(N)$ and $E[(X^{(u,s)})^2] \leq P$. The inner and outer bounding models are capacitated noiseless bit-pipes of capacity $\frac{1}{2}\log(1 + P/N)$.

### E. Gaussian Multiple Access Channel

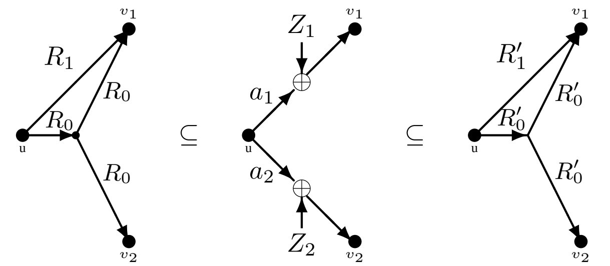
Let $\mathcal{E}_2^m$ be the class of Gaussian multiple access channels. Then each hyperedge $\bar{e} \in \mathcal{E}_2^m$ is a channel

$$(\mathbb{R}^2, p(y^{(v,r)}|x^{(u_1,s_1)}, x^{(u_2,s_2)}), \mathbb{R})$$

with $Y^{(v,r)} = X^{(u_1,s_1)} + X^{(u_2,s_2)} + Z$. Let $E[(X^{(u_1,s_1)})^2] \leq P_1$, $E[(X^{(u_2,s_2)})^2] \leq P_2$, $P_1 \geq P_2$, and $Z \sim \mathcal{N}(0, N)$. Figure 7 shows upper and lower bounding models for the given multiple access channel. For any point in the capacity region of the multiple access channel, there exists a corresponding lower bounding network. The model chosen corresponds to the corner point

$$\begin{aligned} R_2 &= \frac{1}{2}\log\left(1 + \frac{P_2}{N}\right) \\ R_1 &= \frac{1}{2}\log\left(1 + \frac{P_1 + P_2}{N}\right) - \frac{1}{2}\log\left(1 + \frac{P_2}{N}\right). \end{aligned}$$

The upper bounding network is obtained by evaluating the model shown in Figure 4 under the maximizing joint distribution on $(X^{(u_1,s_1)}, X^{(u_2,s_2)})$ and a statistically dependent distortion-$D$ reproduction $U$ of $X^{(u_2,s_2)}$ similar to those used




$R_0 = \frac{1}{2}\log(\frac{(P+N_2/a^2)^2(1-\rho^2)}{(\sqrt{N_2}/a_2-\rho\sqrt{N_1}/a_1)^2P_1+(1-\rho^2)(P+N_2/a_2^2)N_2/a_2^2})$

$R_1 = \frac{1}{2}\log(1 + \frac{P}{N_1/a_1^2}\frac{(\sqrt{N_2}/a_2-\rho\sqrt{N_1}/a_1)^2}{(1-\rho)^2(P+N_2/a_2^2)})$

$R_0' = \frac{1}{2}\log(1 + \frac{P}{N_2/a_2^2})$

$R_1' = \frac{1}{2}\log(1 + \frac{P}{N_1/a_1^2}\frac{(\sqrt{N_2}/a_2-\rho\sqrt{N_1}/a_1)^2}{(1-\rho)^2(P+N_2/a_2^2)})$

Fig. 8. Example models for the Gaussian broadcast channel.

in lossy source coding. Precisely,

$$\begin{aligned} X^{(u_1,s_1)} &= X^{(u_2,s_2)}\sqrt{\frac{P_1}{P_2}} \\ U &= X^{(u_2,s_2)} + Z_2 \\ Z_2 &= -X^{(u_2,s_2)}\frac{D}{P_2} + Z\frac{D}{N\left(1+\sqrt{\frac{P_1}{P_2}}\right)} + W \\ W &\sim \mathcal{N}\left(0, D\left(1 - \frac{D}{P_2} - \frac{D}{N\left(1+\sqrt{\frac{P_1}{P_2}}\right)^2}\right)\right) \\ D &= \frac{P_2N}{P_2+N}. \end{aligned}$$

### F. Gaussian Broadcast Channel

Finally, let $\mathcal{E}_2^b$ be the class of Gaussian broadcast channels. Then each hyperedge $\bar{e} \in \mathcal{E}_2^b$ describes a Gaussian broadcast channel

$$(\mathbb{R}, p(y^{(v_1,r_1)}, y^{(v_2,r_2)}|x^{(u,s)}), \mathbb{R}^2)$$

with $Y^{(v_1,r_1)} = a_1X^{(u,s)} + Z_1$ and $Y^{(v_2,r_2)} = a_2X^{(u,s)} + Z_2$ for some jointly Gaussian random variables $Z_1$ and $Z_2$ with $E[X^2] \leq P$, $E[Z_1^2] = N_1$, $E[Z_2^2] = N_2$, $E[Z_1Z_2] = \rho\sqrt{N_1N_2}$, and $N_1/a_1^2 \leq N_2/a_2^2$. Figure 8 shows an example upper and lower bounding models. The lower bounding model is found by evaluating the broadcast capacity bounds

$$\begin{aligned} R_1 &= \frac{1}{2}\log\left(1 + \frac{(1-\alpha)P}{N_1/a_1^2}\right) \\ R_2 &= \frac{1}{2}\log\left(1 + \frac{\alpha P}{(1-\alpha)P + N_2/a_2^2}\right) \end{aligned}$$

at

$$1 - \alpha = \frac{(\sqrt{N_2}/a_2 - \rho\sqrt{N_1}/a_1)^2}{(1-\rho)^2(a_2^2P + N_2)}.$$

The upper bounding network is obtained by evaluating the model shown in Figure 3.

## IV. Network Capacity Bounds

The remaining task is to investigate the capacity implications of the proposed models. For simplicity, the discussion that follows treats networks of all binary or all Gaussian components. Bounding mixed networks is also straight forward.

For any network $\mathcal{N}$, $\mathscr{R}(\mathcal{N})$ again denotes the full capacity region for network $\mathcal{N}$ – that is the set of rate vectors describing all collections of demands that can be simultaneously met by the given network. Since so little of the capacity region is known, the focus is not on calculating network capacities but on bounding how well the capacity region for the upper bounding network approximates the capacity region of the desired network. The given results bound the accuracy of all capacity bounds that can be obtained on the upper bounding network through analytical or computational means.

The capacity upper bound for a network $\mathcal{N}$ equals the capacity of the network $\mathcal{N}_U$ in which each channel $e \in E$ is replaced by its upper bounding model from Section III. Bounds on the accuracy of $\mathscr{R}(\mathcal{N}_U)$ as an estimate for $\mathscr{R}(\mathcal{N})$ are obtained by comparing $\mathscr{R}(\mathcal{N}_U)$ and $\mathscr{R}(\mathcal{N}_L)$, where $\mathcal{N}_L$ is the network employing the lower bounding models. Results are given in two forms. For any $a \in [0, 1]$, the notation

$$\frac{\mathscr{R}(\mathcal{N})}{\mathscr{R}(\mathcal{N}_U)} \in [a, 1]$$

is used to specify that $\mathcal{R} \in \mathscr{R}(\mathcal{N}_U)$ implies $aR \in \mathscr{R}(\mathcal{N})$, giving a multiplicative bound. For any $b > 0$, the notation

$$\mathscr{R}(\mathcal{N}_U) - \mathscr{R}(\mathcal{N}) \leq b$$

means $\mathcal{R} \in \mathscr{R}(\mathcal{N}_U)$ implies $[\mathcal{R} - b(1, \ldots, 1)]^+ \in \mathscr{R}(\mathcal{N})$.

Where possible, we make statements about complete capacity regions. This is sometimes possible even though the complete capacity regions are unknown for most networks. For example, [3] proves that $\mathscr{R}(\mathcal{N}) = \mathscr{R}(\mathcal{N}_U)$ for all networks $\mathcal{N}$ with $E \subset \mathcal{E}_1^w \cup \mathcal{E}_2^w$.[2] This statement implies, for example, that any collection of unicast demands can be met on network $\mathcal{N}$ if and only if it can be met on network $\mathcal{N}_U$. The result is proved by showing that the upper and lower bounding networks are identical. While this result does not calculate the network capacity, it translates a network capacity problem over noisy channels into a network coding capacity problem over noiseless channels, removing the stochastics and opening the problem to a richer collection of available tools.

In some cases, general statements about bound accuracy can be made only for types of demands where cut-set bounds are tight. This family of problems includes the multicast problem, the multi-source multicast problem, the single-source problem with non-overlapping demands, and the two-resolution single-source multicast problem (see, for example, [6]). Let $\mathscr{R}_c(\mathcal{N})$ be the set of achievable rate vectors corresponding only to demand-types for which cut-sets would be tight if $\mathcal{N}$ were a network of noiseless point-to-point bit-pipes. When the upper and lower bounding models of a component are sufficiently different, analyzing the capacity implications of these differences for all possible demands becomes more difficult. Since bounding cut sets is simpler, we bound the differences between $\mathscr{R}_c(\mathcal{N})$ and $\mathscr{R}_c(\mathcal{N}_U)$ in these cases.

[2]The result in [3] is actually proved for all networks of memoryless, point-to-point channels.

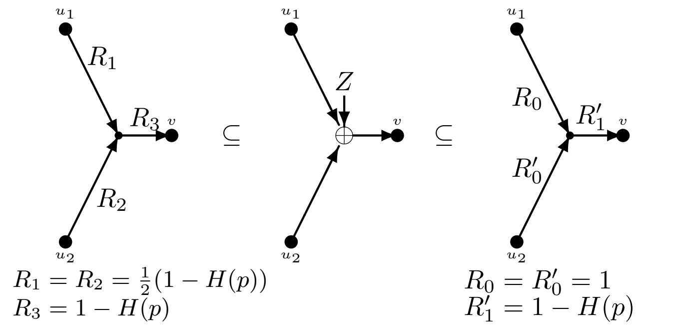



Fig. 9. A variation on the lower bounding model from Figure 5.

### A. Networks of Binary Channels

Consider a network $\mathcal{N}$ of binary channels with $E \subset \mathcal{E}_1^w \cup \mathcal{E}_1^m \cup \mathcal{E}_1^b$. For each channel $e \in E$, define

$$a_e = \begin{cases} 1 & \text{if } e \in \mathcal{E}_1^w \\ \frac{1-H(p_e)}{2} & \text{if } e \in \mathcal{E}_1^m \\ \frac{1-H(p_{e,1} * p_{e,2} * p_{e,1} * p_{e,2})}{1-H(p_{e,1} * p_{e,2})} & \text{if } e \in \mathcal{E}_1^m \text{ \& } Z_{e,1}, Z_{e,2} \text{ ind.} \\ \frac{1-H(p_{e,1} * p_{e,2} * p_{e,2})}{1-H(p_{e,1} * p_{e,2})} & \text{if } e \in \mathcal{E}_1^m \text{ \& } e \text{ stoch. degr.} \end{cases}$$

where $p_e$ is the error probability for $Z$ when $e$ is a multiple access channel and $p_{e,1}$ and $p_{e,2}$ are the error probabilities at the two receivers when $e$ is a broadcast channel.

*Theorem 4.1:* Given a network $\mathcal{N}$ with $E \subset \mathcal{E}_1^w \cup \mathcal{E}_1^m \cup \mathcal{E}_1^b$, let $a^\star = \min_{e \in E} a_e$. Then

$$\frac{\mathscr{R}(\mathcal{N})}{\mathscr{R}(\mathcal{N}_U)} \in [a^\star, 1] \, .$$

*Proof.* The topologies of our upper and lower bounding models match (or can be recast to match – see Figure 9). Since $a^\star$ describes the minimal link-by-link ratio of capacities in lower bounding models to upper bounding models, the following bound applies

$$\mathscr{R}(\mathcal{N}_{LL}) \subseteq \mathscr{R}(\mathcal{N}_L) \subseteq \mathscr{R}(\mathcal{N}) \subset \mathscr{R}(\mathcal{N}_U).$$

Here $\mathcal{N}_{LL}$ is identical to $\mathcal{N}_U$ except that the capacity of each edge has been decreased by factor $a^\star$. Scaling all edges in a network scales the capacity of the network by the same factor, so $\mathscr{R}(\mathcal{N}_L)/\mathscr{R}(\mathcal{N}_U) = a$, which gives the desired result. ■

Theorem 4.1 is tightest when the channel error probabilities for the broadcast and multiple access channels are small. The strength of the given bound is that it doesn't increase with network size. The weakness is that it is limited by its worst-case channel. While additive bounds on cut-sets can also be easily derived using the given results, they seem to be quite loose. Tighter bounds on cut sets can be found by optimizing over the choice of $\alpha$ in each lower bounding model for the multiple access channel. For the broadcast channel, $R_1' - R_1 = 0$ and $R_0' - R_0$ equals $H(p_1 * p_1 * p_2 * p_2) - H(p_1 * p_2)$ when the noise random variables are independent and $H(p_1 * p_2 *$

$p_2) - H(p_1 * p_2)$ when the channel is stochastically degraded. Both values are bounded from above by $0.215$.

### B. Gaussian Channels

Let $\mathcal{N}$ be a network of Gaussian channels with $E \subset \mathcal{E}_2^w \cup \mathcal{E}_2^m \cup \mathcal{E}_2^b$. For simplicity, assume that $P_1 = P_2 = P$ for all multiple access channels and $a_1 = a_2 = 1$ for all broadcast channels. These choices maximize the gap between the upper and lower bounds derived in Section III. For each channel $e \in E$, define

$$b_e = \begin{cases} 0 & \text{if } e \in \mathcal{E}_2^w \\ \frac{1}{2}\log\left(\frac{N_e+4P_e}{N_e+P_e}\right) & \text{if } e \in \mathcal{E}_2^m \\ \frac{1}{2}\log\left(1 + \frac{P_e(\sqrt{N_{e,2}}-\rho_e\sqrt{N_{e,1}})^2}{(1-\rho_e^2)(P_e+N_{e,2})}\right) & \text{if } e \in \mathcal{E}_2^b \end{cases}$$

where $(P_e, N_e)$ are the power constraint and noise variance when $e$ is a multiple access channel and $(P_e, N_{e,1}, N_{e,2}, \rho_e)$ are the power constraint, noise variances, and noise correlation when $e$ is a broadcast channel.

For any network $\mathcal{N}'$ of point-to-point noiseless bit-pipes, and any set $S \subset V$, let $\text{val}(\mathcal{N}', S)$ be the value of cut $S$ defined in the usual way.

*Theorem 4.2:* Given a network $\mathcal{N}$ with $E \subset \mathcal{E}_2^w \cup \mathcal{E}_2^m \cup \mathcal{E}_2^b$, let $b^\star = \sum_e b_e$. Then for all sets $S \subset V$,

$$\min_{\mathcal{N}_U, \mathcal{N}_L} [\text{val}(\mathcal{N}_U, S) - \text{val}(\mathcal{N}_L, S)] \leq \sum_{e: V_1(e) \not\subseteq S^c, V_2(e) \not\subseteq S} b_e.$$

Thus

$$\min_{\mathcal{N}_U} \mathscr{R}_c(\mathcal{N}_U) - \mathscr{R}_c(\mathcal{N}) \in [0, b^\star].$$

Further, when $Z_{e,1}$ and $Z_{e,2}$ are independent for all $e \in \mathcal{E}_2^b$,

$$\min_{\mathcal{N}_U} \mathscr{R}_c(\mathcal{N}_U) - \mathscr{R}_c(\mathcal{N}) \in [0, |E|/2].$$

*Proof.* The results follow from bounding the difference between the upper and lower bounds for each cut. For the multiple access channel, the roles of transmitters 1 and 2 are reversed when necessary to minimize the difference between upper and lower bounds. This assignment of roles for each multiple access channel $e$ is the minimization referred to in the given bounds. ■

Theorem 4.2 generalizes similar bounds by [9] from multicast capacity and unicast capacity to a larger variety of problems where cut-sets are tight on network of noiseless point-to-point bit-pipes and to networks where the noise at two receivers of a broadcast channel is statistically dependent. The result is more restrictive then [9] in its restriction to broadcast and multiple access channels with two receivers.

## V. Conclusion

This work derives additive and multiplicative bounds on the accuracy of the network capacity bounds derived using the network equivalence outer bounds of [3], [4]. For binary channels, constant multiplicative bounds are provided; these bounds apply to all demand types – even types for which analytical capacity solutions are currently unavailable. For Gaussian channels, additive bounds on the cut-set capacities are derived. Since cut-set bounds are not tight in general, a variety of more general approaches to capacity bound calculation have been derived in the literature. These focus primarily on the multiple unicast problem, which is general by [10]. While some of these more careful cut-set approaches apply to noisy point-to-point networks (e.g., [11]), results for broadcast and other channels are not available. Using noiseless bit-pipe upper bounding models allows direct application of all existing techniques to networks comprised of components for which outer bounding network models are available. An example in [12] shows that there exist networks for which the multiple unicast capacity of $\mathcal{N}_U$ equals that of $\mathcal{N}$ while the cut-set bounds on $\mathcal{N}$ are loose by a multiplicative factor that grows linearly with the network size. The example generalizes an example of [2] from noiseless point-to-point links to noisy broadcast channels. For more general networks, bounds on the accuracy of the upper bounding network can be achieved by direct (likely computational) analysis of the upper and lower bounding networks.

## Acknowledgment

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