A Stochastic Control Viewpoint on 'Posterior Matching'-style Feedback Communication Schemes

Todd P. Coleman
ECE Department
Coordinated Science Laboratory
University of Illinois
colemant@illinois.edu

***Abstract*—This paper re-visits Shayevitz & Feder's recent 'Posterior Matching Scheme', a deterministic, recursive, capacity-achieving feedback encoding scheme for memoryless channels. We here consider the feedback encoder design problem from a stochastic control perspective. The state of the system is the posterior distribution of the message given current outputs of the channel. The per-trial reward is the average 'reduction in distance' of the posterior to the target unit step function. We show that the converse to the channel coding theorem with feedback upper bounds the optimal reward, and that the posterior matching scheme is an optimal policy. We illustrate that this 'reduction in distance' symbolism leads to the existence of a Lyapunov function on the Markov chain under this optimal policy, which leads to demonstration of achievability for all rates less than capacity.**

I. Introduction

This paper re-visits Shayevitz & Feder's recent 'Posterior Matching Scheme' [1], [2], which is an explicit, recursive feedback encoding scheme that achieves capacity on memoryless channels. Analogous to [2], we treat the state of the system as the posterior cumulative distribution function (PCDF) of the message given current outputs of the channel. However, instead of thinking in terms of iterated function systems and contraction mappings, we take a stochastic control and Lyapunov function viewpoint. The per-trial reward is the 'reduction in distance' of the PCDF to the target unit step function. We combine this with the converse to the channel coding theorem with feedback to bound the optimal reward, and show that the posterior matching (PM) scheme is an optimal policy. Lastly, we illustrate that this 'reduction in distance' of the PCDF symbolism leads to the existence of a simple Lyapunov-style drift inequality on the Markov chain under this optimal policy. Using this drift condition and successively finer quantizations of the $[0, 1)$ interval, we show that this condition suffices to achieve capacity. The stochastic control perspective also provides a canonical encoder design approach to feedback communication problems more generally.

Since the KL distance plays a fundamental role here, the Lyapunov function and its drift will most likely allow for a succinct way to describe error exponents [1]. Lastly, this methodology has the potential to provide a canonical recursive encoder design methodology for more complicated settings, including:

- Noisy feedback encoding using a Partially Observed Markov Decision Process (POMDP) framework [3];
- Feedback encoding over non-memoryless channels using the converse with the directed information; and
- Feedback encoding for multiterminal problems by identifying tightness conditions on their converse theorems.

A. Our Contribution

This approach differs here from [1], [2] in a couple of ways. First, a (perhaps obvious to the original authors in [1], [2]) simple observation is made from the converse to the channel coding theorem with feedback, and tightness conditions on the bounds - which provide an impetus as to how the PM scheme arises as an encoder design principle. Our approach provides a unified approach to develop both a) a reward function that has a communication interpretation for the control problem and b) a Lyapunov function on the induced Markov chain to directly show achievability of any rate $R < C$ (as compared to the iterated function systems and contraction mapping necessary conditions discussed in [2]). Both a) and b) are established with the 'reduction in Kullback-Leibler distance from the posterior to the target message point' symbolism.

By embedding the problem within this framework, we illustrate an interesting symbiotic relationship between stochastic control and information theory. Information theory aids control in that simple information-theoretic inequalities -with conditions for tightness - bound the optimal average cost, so that Bellman's equation need not be explicitly solved. Control aids information theory in that we can explicitly show achievability of any rate less than capacity because of illustrating a stochastic Lyapunov function (which pertains to a reduction in Kullback-Leibler distance from the posterior to the target message point).

II. Preliminaries

Consider a memoryless channel with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. The conditional distribution pertaining to the memoryless channel is given by $\mathbb{P}_{Y|X}(\cdot|x)$. Let W be a random message point that is uniformly distributed on $[0, 1)$.

[1]an approach to deriving such exponents for the PM scheme has been initially undertaken in [2], although sometimes strictly below the mutual information

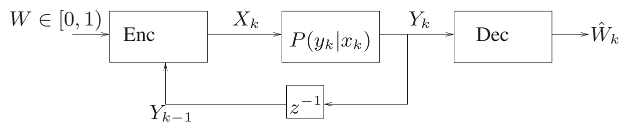

Fig. 1. General Communication with Feedback

Throughout this discussion we assume that $\mathcal{X}$ has an ordering so that the cumulative distribution function $F_X(x)$ is well-defined for any $x \in \mathcal{X}$. So we take our message W and transform it to the channel input X_n at each time unit n. We assume we have instantaneous feedback at time n of the previous channel outputs not including the current one transmitted: Y^{n-1}. Now that we have feedback, although capacity is not increased, the complexity of encoding and decoding, and the error probability performance, can be significantly improved. Examples of the feedback coding with a continuous W uniformly distributed on $[0,1)$ include Horstein's scheme on the BSC [4] and Schalkwijk/Kailaith's scheme [5] on the AWGN channel. That being said, there was no unifying theme linking these schemes, nor an approach for an arbitrary memoryless channel, before the Shayevitz-Feder PM scheme [1], [2] - which unifies all previous approaches with a simple recursive interpretation.

A. Notations

- For a communication channel with transition law $\mathbb{P}_{Y|X}(\cdot|x)$, define its capacity C and define $\mathbb{P}_X(\cdot)\,(F_X(\cdot))$ as the probability distribution (CDF) on the input that achieves C.
- A transmission scheme with feedback is a sequence of measurable transmission functions $\left\{g_n : [0,1) \times \mathcal{Y}^{n-1} \to \mathcal{X}\right\}$ so that the input to the at time n is given by

$$X_{n+1} = g_{n+1}(W, Y^n). \tag{1}$$

- Denote $\mathfrak{F}$ to be the space of all cumulative distribution functions (CDF)s on $[0,1)$ - i.e. the space of all monotone, surjective non-decreasing functions $F : [0,1) \to [0,1)$. Denote $\mathfrak{F}^* \subset \mathfrak{F}$ to be the set of all unit-step functions.
- Denote $F_w^* \in \mathfrak{F}^*$ to be the step function with jump at w.
- Denote the posterior CDF of W at time n as

$$F_n(\cdot) \triangleq F_{W|Y^n}(\cdot|Y^n)$$

- Denote $\Delta \triangleq 2^{-R}$ with $R \in (0, C)$.
- For a probability distribution Q on $[0,1)$, define $\langle Q \rangle_n$ to be the probability mass function created by truncating Q to 2^{nR} equal-length intervals of length $\Delta^n = 2^{-nR}$:

$$\langle Q \rangle_n(j) = \int_{(j-1)\Delta^n}^{j\Delta^n} Q(dx), \quad 1 \leq j \leq 2^{nR} \tag{2}$$

See Figure 2.
- For a message point $w \in [0,1)$, define $j(w,n)$ to be the index of the cell containing w amongst the 2^{nR} cells, each of width $\Delta^n = 2^{-nR}$:

$$j(w,n) \triangleq \lfloor w 2^{nR} \rfloor + 1 \tag{3}$$

- A recursive scheme admits an achievable rate R if,

$$\mathbb{P}\left(j(W,n) \neq j(\hat{W}_n, n)\right) \to 0 \tag{4}$$

- For a Markov process $\{S_1, S_2, \ldots\}$ defined with state space $\mathcal{S}$, a sequence of functions $\{V_n : \mathcal{S} \to \mathbb{R}_+\}$ will be termed a set of time-varying *Lyapunov functions* [6] if there exists a drift $D > 0$ s.t. for any $s \in \mathcal{S}$ satisfying $V_n(s) > 0$, there holds

$$\mathbb{E}\left[V_{n+1}(S_{n+1})\,|S_n = s\right] - V_n(s) \leq -D \tag{5a}$$

This can be succinctly stated as

$$\mathbb{E}\left[V_{n+1}(S_{n+1})\,|S_n = s\right] - V_n(s) \leq -D 1_{\{V_n(s)>0\}}.$$

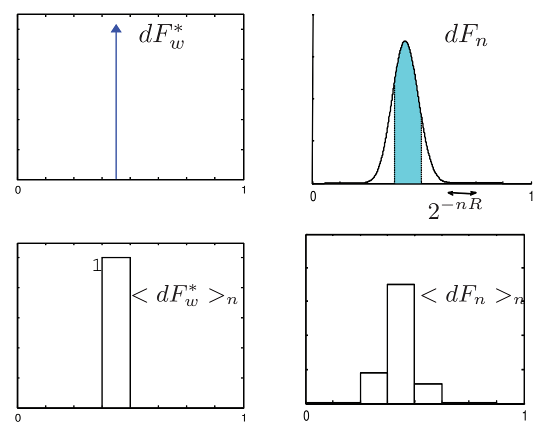

Fig. 2. An example of the distributions F_w^* and F_n on $[0,1)$ discretized to pmfs $< dF_w^* >_n$ and $< dF_n >_n$ on $\{1, \ldots, 2^{nR}\}$.

B. The Posterior Matching Scheme

As recently discussed in [1], [2], for general memoryless communication channels, a recursive capacity-achieving coding scheme is given by

$$X_{n+1} = g_{n+1}(W, Y^n) \quad = \quad F_X^{-1}(F_n(W)). \tag{6}$$

The high-level idea is that $F_n(W)$ is uniformly distributed on $[0,1)$ regardless of Y^n, and so it follows that [1, Sec. II]

- X_{n+1} is independent of Y^n and so due to the channel being memoryless, Y_{n+1} is independent of Y^n.
- The marginal distribution on X_{n+1} is $\mathbb{P}_X(\cdot)$, the capacity-achieving distribution.

Other nice properties of the PM feedback scheme include:

- There is no pre-specified rate or block length.
- There is no forward error correction - it simply adapts on the fly.
- The scheme admits a simple recursive structure [2].

[2]whereby in many cases, a sufficient statistic (i.e. the median of the posterior in the BSC case [4] or the linear MMSE estimate of the first transmission in the AWGN case [5]) can be fed back to the encoder

These properties make this class of feedback encoder designs amenable to a number of applications, including control over noisy channels [7], as well as principled design approaches for brain-machine interfaces [8].

C. A Perspective from Shannon's Converse with Feedback

By revisiting Shannon's converse to the the channel coding with feedback [9], note that

$$\begin{aligned}\frac{1}{n}I(W;Y^n) &= \frac{1}{n}H(Y^n) - \frac{1}{n}H(Y^n|W) \\ &= \frac{1}{n}H(Y^n) - \sum_{i=1}^{n}\frac{1}{n}H(Y_i|Y^{i-1},W) \\ &\le \sum_{i=1}^{n}\frac{1}{n}H(Y_i) - \frac{1}{n}H(Y_i|Y^{i-1},W) \quad (7) \\ &= \sum_{i=1}^{n}\frac{1}{n}H(Y_i) - \frac{1}{n}H(Y_i|X_i,Y^{i-1},W) \quad (8) \\ &= \sum_{i=1}^{n}\frac{1}{n}H(Y_i) - \frac{1}{n}H(Y_i|X_i) \quad (9) \\ &\le C. \quad (10)\end{aligned}$$

where (7) follows because conditioning reduces entropy; (8) follows from (1); and (9) follows because the channel is memoryless. Note that:

- (7) holds with equality under the PM scheme: X_{i+1} is independent of Y^i.
- (10) holds with equality under the PM scheme: X_i is drawn according to $\mathbb{P}_X(\cdot)$.

III. A STOCHASTIC CONTROL APPROACH

Now we note that for a general communication scheme (1), all the Y^ns are potentially required to generate the input, resulting in an explosion in the state of the system. Note that from above, a sufficient statistic for recursive encoding appears to be the posterior distribution F_n. We also observed from the previous section that the PM scheme is an optimal iterative procedure w.r.t. maximizing $\frac{1}{n}I(W;Y^n)$ for any n. A natural question to ponder is if it can be cast from a stochastic control framework. In some sense, the encoder, with feedback, is recursively attempting to *steer* the posterior distribution F_n in the direction of F_w^*. We now show that one interpretation of that is as follows:

$$\begin{aligned}D(F_w^*\|F_n) - D(F_w^*\|F_{n+1}) &= \int_{x=0}^{1}\delta(x-w)\log\frac{\delta(x-w)}{dF_n(x)} \\ &\quad - \int_{x=0}^{1}\delta(x-w)\log\frac{\delta(x-w)}{dF_{n+1}(x)} \\ &= \log\frac{dF_{n+1}(w)}{dF_n(w)} \quad (11a)\end{aligned}$$

Thus, we see that the reduction in distance from the target F_w^* and the current state F_n to the distance from F_w^* and F_{n+1} is given by the log likelihood ratio.

A. A Stochastic Control Viewpoint

Consider Figure 1. Feedback Y^{i-1} is passed to the encoder. From this, the posterior F_{i-1} is formed. And this is used, along with the true message W, to specify the next channel input:

$$X_i = u_i(W, F_{i-1}) \quad (12)$$

where $u_i : [0,1) \times \mathfrak{F} \to \mathcal{X}$ is a Borel-measurable function. At each time i, given F_{i-1}, our objective is to specify a control u_i to maximize the expected reduction in KL distance between F_n and F_W^*. This in turn equates to maximizing $\frac{1}{n}I(W;Y^n)$.

- **State Definition** Define the state of the system at time i to be the posterior CDF at time i, $S_i = F_i \in \mathfrak{F}$.
- **Control Policy** Define the control policy γ_i for the control input u_i to be of the form: $u_i = \gamma_i(u^{i-1}, s^{i-1})$, where $u_i : [0,1) \times \mathfrak{F} \to \mathcal{X}$.
- **Reward Definition** The reward attained at state $s \in \mathfrak{F}$ with control input $u \in \mathcal{U}$ is given by:

$$\begin{aligned}g(u,s) &\triangleq \mathbb{E}\left[D(F_W^*\|F_n) - D(F_W^*\|F_{n+1})\,|S_n = f, U_n = u\right] \\ &= \mathbb{E}\left[\log\frac{dF_{n+1}(W)}{dF_n(W)}|F_n = f, U_{n+1} = u\right] \quad (13)\end{aligned}$$

which is the expected reduction in KL distance at the next time step from the current state f to the target F_w^*.

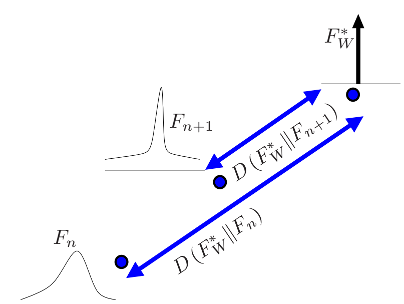

Fig. 3. The reward function as the reduction in 'distance' of the posterior from time n to time $n+1$ from certainty

- **State Transition Equation** Suppose for brevity that the channel is continuous with a conditional PDF, and that the F_{i-1} has an associated PDF given by dF_{i-1}. Then note that from Bayes' rule, we have:

$$\begin{aligned}dF_i(\cdot) &= \frac{f_{Y|X}(y_i|, u_i(\cdot, F_{i-1}))\,dF_{i-1}(\cdot)}{\int_{w'=0}^{1} f_{Y|X}(y_i|, u_i(w', dF_{i-1}))\,dF_{i-1}(w')} \\ &= \Upsilon(F_{i-1}, y_i, u_i) \quad (14)\end{aligned}$$

given by Υ in (14), the channel law $f_{Y|X}(\cdot|x)$, and the control input u. Thus this is a controlled Markov chain. See Figure 4.

Lemma 3.1: The PM scheme is an optimal solution to the stochastic control problem above.

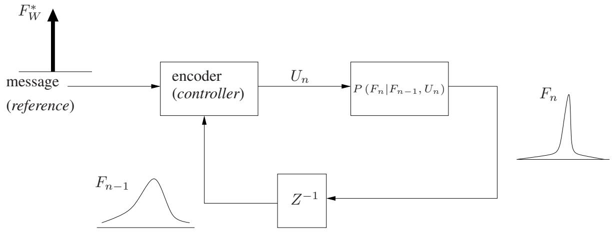

Fig. 4. The evolution of the posterior as a controlled Markov chain.

Proof: From our reward definition, the time-average reward is given by

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} g(U_i,S_i)\right] &= \frac{1}{n}\sum_{i=1}^{n} I(W;Y_i|F_{i-1}) \\ &= \frac{1}{n}\sum_{i=1}^{n} I(W;Y_i|Y^{i-1}) \\ &= \frac{1}{n} I(W;Y^n)\end{aligned}$$

And so from Section II-C on the converse to the channel coding theorem with feedback, it follows that the PM scheme is an optimal policy for any n. ∎

IV. A Lyapunov Function approach to proving Achievability

In the below discussion we illustrate that the Markov chain $\{S_n = F_n \in \mathfrak{F}\}$ induced by any optimal scheme achieves capacity - by virtue of a Lyapunov function. Note that:

$$g(u^*(f),f) = I(W;Y_{n+1}|F_n=f) = C1_{\{f\neq F_W^*\}} \quad (15)$$

where the indicator function holds for the case when the posterior is the point mass corresponding to the message, for which future log likelihood ratios will be 0. Define

$$V_\infty(f) \triangleq \int_{w=0}^{1} D\left(F_w^* \| f\right) df(w) \quad (16)$$

and note that

$$\begin{aligned}&\mathbb{E}\left[V_\infty(F_{n+1})|F_n=f\right] - V_\infty(f) \\ &= \int_{f'\in\mathfrak{F}} \left[\int_{w=0}^{1} D\left(F_w^*\|f'\right) df'(w)\right] \mathbb{P}_{F_{n+1}|F_n}\left(f'|f\right) df' \\ &- \int_{w=0}^{1} D\left(F_w^*\|f\right) df(w) \\ &= \mathbb{E}\left[D\left(F_W^*\|F_{n+1}\right) - D\left(F_W^*\|F_n\right)|F_n=f\right] \quad (17a) \\ &= \mathbb{E}\left[\log\frac{dF_n(W)}{dF_{n+1}(W)}\Big|F_n=f\right] \quad (17b) \\ &= -\,\mathbb{E}\left[D\left(F_{n+1}\|F_n\right)|F_n=f\right] \\ &= -\,I\left(W;Y_{n+1}|F_n=f\right) \\ &= -\,C1_{\{V_\infty(f)>0\}} \quad (17c)\end{aligned}$$

where (17b) follows from (11); and (17c) follows because, if $f = F_w^*$, there is no subsequent information gain - the likelihood ratio is a.s. 0.

It appears as if from from (17), (5), and the non-negativity of the KL divergence that the function $V_\infty : \mathfrak{F} \to \mathbb{R}_+$ given in (16) is a Lyapunov function with drift C. However, for any $f \notin \mathfrak{F}^*$, $V_\infty(f) = \infty$. We now follow that intuition and eliminate this drawback by using finite time-varying functions $\{V_n\}$ that approximate V_∞, which follows from an asymptotic partition formulation of the KL divergence.

- For any $f \in \mathfrak{F}$, we can define, in analogy with (16):

$$\begin{aligned}V_n(f) &\triangleq \int_{w=0}^{1} D\left(\langle F_w^*\rangle_n \,\|\, \langle f\rangle_n\right) df(w) \quad (18a)\\ &= \mathbb{E}\left[-\log\langle F_n\rangle_n\left(j(W,n)\right)|F_n=f\right] \quad (18b)\\ &= H\left(\langle F_n\rangle_n|F_n=f\right)\end{aligned}$$

- Since the partitions of $[0,1)$ into length 2^{-nR} intervals correspond to an asymptotically generating sequence of sub-σ-fields of $[0,1)$, the KL distance between two probability measures with CDFs F and F', can be expressed as [10, Corollary 5.2.3]:

$$D\left(\langle F\rangle_n \,\|\, \langle F'\rangle_n\right) \uparrow D\left(F\|F'\right). \quad (19)$$

- Define $\epsilon_n(f)$ as follows:

$$\epsilon_n(f) = C1_{\{V_n(f)>0\}} - \mathbb{E}\left[D\left(\langle F_{n+1}\rangle_n \,\|\, \langle F_n\rangle_n\right)|F_n=f\right] \quad (20)$$

- Define the open [3] set $\mathcal{O}_n \subset \mathfrak{F}$ as

$$\mathcal{O}_n = \{f\in\mathfrak{F} : V_n(f)>0\}.$$

We now enumerate the following two technical conditions that we will assume throughout:

Assumption 1:
• a) The kernel $\mathbb{P}_{F_{n+1}|F_n}(\cdot|f)$ induced by policy γ has a Feller property [6], in that

$$\int_{f'\in\mathfrak{F}} D\left(\langle f'\rangle_n \,\|\, \langle f\rangle_n\right) \mathbb{P}_{F_{n+1}|F_n}\left(df'|f\right)$$

is continuous in f on $\mathcal{O}_n$.
• b) $I(W;Y_{n+1}|F_n=f) = C1_{\{V_\infty(f)>0\}}$ for all $f\in\mathfrak{F}$.
Note that under the PM scheme, a) and b) are both satisfied for well-behaved channels. This leads to:

Lemma 4.1: For any feedback coding scheme and communication channel satisfying Assumption 1, $\epsilon_n(f) \downarrow 0$ uniformly in f over $\mathcal{O}_n$.

Proof:

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[D\left(\langle F_{n+1}\rangle_n \,\|\, \langle F_n\rangle_n\right)|F_n=f\right] \\ &= \mathbb{E}\left[\lim_{n\to\infty} D\left(\langle F_{n+1}\rangle_n \,\|\, \langle F_n\rangle_n\right)|F_n=f\right] \quad (21)\\ &= \mathbb{E}\left[D\left(F_{n+1}\|F_n\right)|F_n=f\right] \quad (22)\\ &= C1_{\{V_\infty(f)>0\}}. \quad (23)\end{aligned}$$

where (21) follows from the dominated convergence theorem and (19); and (22) follows from (19). Since $1_{\{V_n(f)>0\}} \downarrow$

[3] $\mathfrak{F}$ is a compact topological space, endowed with the weak topology and metrized by the Prohorov distance [11].

$1_{\{V_\infty(f)>0\}}$, and since (19) holds, it follows that $\epsilon_n(f) \downarrow 0$ in n. Note from from Assumption 1a and (20) that $\epsilon_n(f)$ is continuous in f on $\mathcal{O}_n$. So from Dini's theorem [12], since $\mathcal{O}_n \subset \mathfrak{F}$ and $\mathfrak{F}$ is compact, $\epsilon_n(f) \downarrow 0$ uniformly in $\mathcal{O}_n$. ■

Theorem 4.2: Any feedback coding scheme consisting of mappings $X_{n+1} = e_{n+1}(W, F_n)$ over a memoryless channel with induced transition law $\mathbb{P}_{F_{n+1}|F_n}(\cdot|f)$ that satisfies Assumption 1 achieves all rates $R < C$.

Proof:

- For any $\epsilon > 0$, there exists an $N_0(\epsilon)$ s.t. for all $n > N_0(\epsilon)$ and all $f \in \mathcal{O}_n$,

$$\begin{aligned} &\mathbb{E}\left[V_n(F_{n+1}) - V_n(F_n)|F_n = f\right] &\text{(24a)}\\ &= -\mathbb{E}\left[D\left(\langle F_{n+1}\rangle_n \,\|\, \langle F_n\rangle_n\right)|F_n = f\right] \\ &\leq -(C-\epsilon)1_{\{V_n(f)>0\}} &\text{(24b)}\end{aligned}$$

  where (24b) follows from (20) and Lemma 4.1.

- Next, assuming that $\epsilon \in (0, C-R)$, we have:

$$\begin{aligned} &\mathbb{E}\left[V_{n+1}(F_{n+1}) - V_n(F_n)|F_n = f\right] \\ =&\mathbb{E}\left[V_n(F_{n+1}) - V_n(F_n)|F_n = f\right] \\ +&\mathbb{E}\left[V_{n+1}(F_{n+1}) - V_n(F_{n+1})|F_n = f\right] \\ \leq& -(C-\epsilon)1_{\{V_n(f)>0\}} &\text{(25)}\\ +&\mathbb{E}\left[V_{n+1}(F_{n+1}) - V_n(F_{n+1})|F_n = f\right] \\ =& -(C-\epsilon)1_{\{V_n(f)>0\}} \\ -&\mathbb{E}\left[\log\frac{\langle F_{n+1}\rangle_{n+1}(j(W,n+1))}{\langle F_{n+1}\rangle_n(j(W,n))}|F_n = f\right] &\text{(26)}\\ \leq& -(C-R-\epsilon)1_{\{V_n(f)>0\}} &\text{(27)}\end{aligned}$$

  where (25) follows from (24); (27) follows by careful consideration of the definition of $j(W,n)$: $\frac{\langle F_{n+1}\rangle_{n+1}(j(W,n+1))}{\langle F_{n+1}\rangle_n(j(W,n))}$ corresponds to the probability mass function of a discrete random variable of cardinality 2^R, and thus (26) is its entropy - which is at most R.

- With this, we have:

$$\begin{aligned} &\mathbb{E}\left[V_{n+M}(F_{n+M}) - V_n(F_n)|F_n = f\right] \\ =& \sum_{k=n}^{n+M-1}\mathbb{E}\left[V_{k+1}(F_{k+1}) - V_k(F_k)|F_n = f\right] \\ =& \sum_{k=n}^{n+M-1}\mathbb{E}\left[\mathbb{E}\left[V_{k+1}(F_{k+1}) - V_k(F_k)|F_k\right]|F_n = f\right] \\ \leq& \sum_{k=n}^{n+M-1}\mathbb{E}\left[-(C-R-\epsilon)1_{\{V_k(F_k)>0\}}|F_n = f\right] \\ =& \sum_{k=n}^{n+M-1}\left[-(C-R-\epsilon)\mathbb{P}\left(V_k(F_k)>0|F_n = f\right)\right]\end{aligned}$$

Note that the above sum is strictly non-increasing and thus has a limit. Since $\mathbb{E}\left[V_{n+M}(F_{n+M})|F_n = f\right] \geq 0$ for any M, and thus as $M \to \infty$, it follows that:

$$\begin{aligned} &\lim_{M\to\infty}\mathbb{P}\left(V_{n+M}(F_{n+M})>0|F_n = f\right) = 0. &\text{(28)}\\ &\Rightarrow \lim_{M\to\infty}\mathbb{P}\left(V_{n+M}(F_{n+M})>0\right) \\ &= \lim_{M\to\infty}\mathbb{E}\left[\mathbb{P}\left(V_{n+M}(F_{n+M})>0|F_n\right)\right] \\ &=\mathbb{E}\left[\lim_{M\to\infty}\mathbb{P}\left(V_{n+M}(F_{n+M})>0|F_n\right)\right] &\text{(29)}\\ &=0 &\text{(30)}\end{aligned}$$

where (29) follows from the bounded convergence theorem and (30) follows from (28). Thus $V_n \to 0$ in probability, or equivalently from (18b), $\langle F_n\rangle_n(j(W,n)) \to 1$ in probability. From (4), it follows that $R < C$ is achievable.

■

V. Conclusion

This Kullback-Leibler based Lyapunov function approach will most likely lead to a simple way to also understand error exponents. By identifying equality conditions in the converse, and by including the *posterior* in the state space, we can directly get to statements about achievability. Lastly, extending this converse-oriented and stochastic control approach to problems with noisy feedback, non-memoryless channels, and multiple terminals will be the subject of future work.

VI. Acknowledgements

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