

Competitive Scheduling in Wireless Collision Channels with Correlated Channel State

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Abstract

We consider a wireless collision channel, shared by a finite number of mobile users who transmit to a common base station. Each user wishes to optimize its individual network utility that incorporates a natural tradeoff between throughput and power. The channel quality of every user is affected by global and time-varying conditions at the base station, which are manifested to all users in the form of a common channel state. Assuming that all users employ stationary, state-dependent transmission strategies, we investigate the properties of the Nash equilibrium of the resulting game between users. While the equilibrium performance can be arbitrarily bad (in terms of aggregate utility), we bound the efficiency loss at the best equilibrium as a function of a technology-related parameter. Under further assumptions, we show that sequential best-response dynamics converge to an equilibrium point in *finite* time, and examine how to exploit this property for better network usage.

1 Introduction

Wireless technologies are broadly used today for both data and voice communications. The transmission protocols of wireless devices need to cope with the scarce resources available, such as bandwidth and energy. Additional difficulties relate to the *dynamic* nature of wireless networks. For example, the mobility of terminals and the frequent change in their population introduce new challenges for routing protocols. An additional distinctive dynamic feature of wireless communications is the possible time variation in the channel quality between the sender and the receiver, an effect known as channel fading [7].

Motivated by scalability considerations, it has been advocated in recent years that mobiles should have the freedom to distributively adjust their transmission parameters (e.g., [10]). This leads to distributed network domains, in which situation-aware users take autonomous decisions with regard to their network usage, based on the current network conditions and their individual preferences. As users generally do not coordinate their network actions, non-cooperative game-theory has become a natural theoretical framework for analysis and management (see, e.g., [3, 1, 8], and [10] for a survey). The general framework that will be considered here is that of users who obtain state information and may accordingly control their transmission parameters. A major issue in this context is whether such information is well exploited by self-interested users, or rather misused.

In this paper, we consider a wireless collision channel, shared by a finite number of users who wish to optimally schedule their transmissions based on a natural tradeoff between throughput and power (proportional to the number of transmissions). The channel quality between each user and the base station is randomly time-varying and observed by the user prior to each transmission decision.

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The bulk of the research in the area has been carried under a simplified assumption that the channel state processes of different users are independent (see e.g., [4, 9]). In practice, however, there are global system effects, which simultaneously affect the quality of all transmissions (e.g., thermal noise at the base station, or common weather conditions). Consequently, a distinctive feature of our model is that the state processes of different users are *correlated*. As an approximating model, we consider in this paper the case of *full* correlation, meaning that all users observe the same state prior to transmission. A fully correlated state can have a positive role of a coordinating signal, in the sense that different states can be “divided” between different users. On the other hand, such state correlation increases the potential deterioration in system performance due to non-cooperation, as users might transmit simultaneously when good channel conditions are available.

Our result indicate that both the positive and negative effects of state correlation are possible. User interaction and unwillingness to give up the better-quality states leads to arbitrarily bad equilibria in terms of the aggregate utility. At the same time, there exist good-quality equilibria, whose performance-gap from the social welfare solution can be bounded as a function of a technology-related parameter. In some special cases, we establish that the underlying scheduling game is a potential game [11]. We use this property and additional structure in our game, to establish the *finite-time* convergence of sequential best-response dynamics (although the action space is continuous). We further show by simulations that convergence to an equilibrium can be very fast. We use the latter property to indicate how good-quality equilibria can be iteratively obtained. We also extend the potential game results in the literature by providing some necessary and sufficient conditions for existence of an ordinal potential in games.

The structure of the paper is as follows. The model and some basic properties thereof are presented in Section 2. The social welfare problem is defined and characterized in Section 3. We then proceed in Section 4 to study the efficiency loss under selfish user behavior, as compared to the social welfare operating point. Section 5 studies the convergence properties of best-response dynamics and their significance. In section 6 we provide conditions for existence of ordinal potential in games. Conclusions are drawn in Section 7.

2 The Model and Preliminaries

We consider a wireless network, shared by a finite set of mobile users $\mathcal{M} = \{1, \dots, M\}$ who transmit at a fixed power level to a common base station over a shared collision channel. Time is slotted, so that each transmission attempt takes place within slot boundaries that are common to all. Transmission of a user is successful only if no other user attempts transmission simultaneously. Thus, at each time slot, at most one user can successfully transmit to the base station. To further specify our model, we start with a description of the channel between each user and the base station (Section 2.1), ignoring the possibility of collisions. In Section 2.2, we formalize the user objective and formulate the non-cooperative game which arises in a multi-user shared network.

2.1 The Physical Network Model

Our model for the channel between each mobile (or *user*) and the base station is characterized by two basic elements.

a. Channel state process. We assume that the channel state between mobile m and the base station evolves as a stationary process $H^m(t)$, $t \in \mathbb{Z}_+$ (e.g., Markovian) taking values in a set $\mathcal{H}^m = (1, 2, \dots, h^m)$ of h^m states. The stationary probability that mobile m would observe state $i \in \mathcal{H}^m$ at any time t is given by π_i^m .

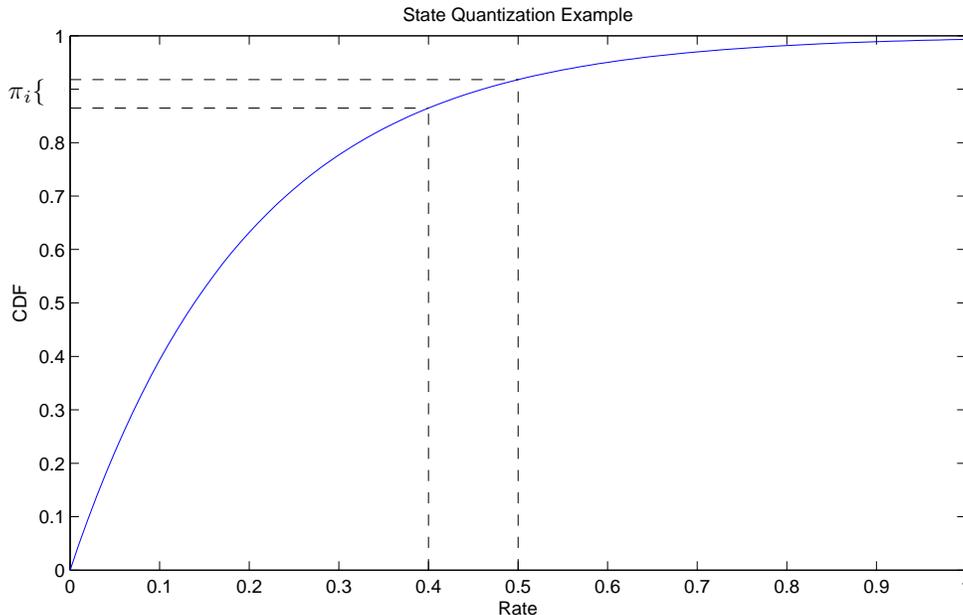


Figure 1: State Quantization Example: Rates between 0.4 and 0.5 are represented by a single state i , with steady state probability of π_i . Both π_i and corresponding expected rate can be calculated from the cumulative density function (CDF) of the underlying rate process

b. Expected data rate. We denote by $R_i^m > 0$ the expected data rate (or simply, the rate)¹ that user m can sustain at any given slot as a function of the current state $i \in \mathcal{H}^m$. We further denote by $\mathcal{R}^m = \{R_1^m, R_2^m \dots, R_{h^m}^m\}$ the set of all data rates for user m , and define $\mathcal{R} = \mathcal{R}^1 \times \mathcal{R}^2 \dots \times \mathcal{R}^M$. For convenience, we assume that for every $m \in \mathcal{M}$ the expected data rate R_i^m strictly decreases in the state index i , so that $R_1^m > R_2^m \dots > R_{h^m}^m$, i.e., state 1 represents the “best state” in which the highest rate can be achieved.

Note that the actual channel quality may still take continuous values, which each user reasonably quantifies into a finite number of information states. Using the cumulative density function of the underlying channel quality the expected data rates and their associated steady state probabilities can be obtained as in figure 1. The motivation behind considering a discrete state process rather than the actual channel quality is the technical inability of mobiles to sense and process a continuum of channel quality information.

A central assumption in this paper is that the state processes of different users are fully correlated, as we formalize below.

Assumption 1 (Full Correlation). *All users observe the same channel state $H(t)$ in any given time t . That is, for every mobile $m \in \mathcal{M}$: (i) $\mathcal{H}^m = \mathcal{H} = \{1, 2, \dots, h\}$, (ii) $\pi_i^m = \pi_i$ for every $i \in \mathcal{H}$ and (iii) $H^m(t) = H(t)$ (where \mathcal{H} is the common state space, and $\pi = (\pi_1, \dots, \pi_h)$ is its stationary distribution).*

We emphasize that although all mobiles observe the same state, the corresponding rates R_i^m need *not* be equal across mobiles, i.e., in our general model we do *not* assume that $R_i^m = R_i^k$, $m, k \in \mathcal{M}$, $i \in \mathcal{H}$. The case where the latter condition does hold will be considered as a special case in Section 5.

¹Say, in bits per second.

The above model can be used to capture the following network scenario. The channel state corresponds to *global* conditions that affect all user transmissions. Examples may include thermal noise at the base station and weather conditions (that play a central role, e.g., in satellite networks), which affect all mobiles in a similar manner. The state information would be measured by the mobiles. After observing the state at the beginning of each slot, a user may respond by adjusting its coding scheme in order to maximize its data rate on that slot. The rate R_i^m thus takes into account the quality of the current state i , the coding scheme adapted by the user, and “local” characteristics, such as the user’s transmission power and location relative to the base station. We note that R_i^m is an *average* quantity, which averages possible fluctuations in local characteristics.

2.2 User Objective and Game Formulation

In this subsection we describe the user objective and the non-cooperative game which arises as a consequence of the user interaction over the collision channel. In addition, we provide some basic properties and examples for the Nash equilibrium of the underlying game.

2.2.1 Basic Definitions

The basic assumption of our model is that users always have packets to send, yet they are free to determine their own transmission schedule in order to fulfill their objectives. Furthermore, users are unable to coordinate their transmission decisions.

Our focus in this paper is on *stationary* transmission strategies, in which the decision whether to transmit or not can depend (only) on the current state. A formal definition is provided below.

Definition 1 (Stationary Strategies). *A stationary strategy for user m is a mapping $\sigma^m : \mathcal{H} \rightarrow [0, 1]^h$. Equivalently, σ^m is represented by an h -dimensional vector $\mathbf{p}^m = (p_1^m, \dots, p_h^m) \in [0, 1]^h$, where the i -th entry corresponds to user m ’s transmission probability when the observed state is i .*

We use the term multi-strategy when referring to a collection of user strategies, and denote by the vector $\mathbf{p} = \{\mathbf{p}^1, \dots, \mathbf{p}^M\}$ the multi-strategy comprised of all users’ strategies. The multi-strategy representing strategies of all users but the m th one is denoted by \mathbf{p}^{-m} . For a given multi-strategy \mathbf{p} , we define below the Quality of Service (QoS) measures that determine user performance. Let W^m be the (fixed) transmission power of user m per transmission attempt, and denote by $\tilde{P}^m(\mathbf{p}^m)$ its average power investment, as determined by its strategy \mathbf{p}^m . Then clearly, $\tilde{P}^m(\mathbf{p}^m) = W^m \sum_{i=1}^h \pi_i p_i^m$ for every user m . We normalize the latter measure by dividing it by W^m , and consider henceforth the normalized power investment, given by

$$P^m(\mathbf{p}^m) = \sum_{i=1}^h \pi_i p_i^m. \quad (1)$$

For simplicity, we shall refer to $P^m(\mathbf{p}^m)$ as the power investment of user m . We assume that each user m is subject to an individual power constraint $0 < \bar{P}^m \leq 1$, so that any user strategy \mathbf{p}^m should obey

$$P^m(\mathbf{p}^m) \leq \bar{P}^m. \quad (2)$$

The vector of power constraints is denoted by $\bar{\mathbf{P}} = (\bar{P}^1, \dots, \bar{P}^M)$.

The second measure of interest is the mobile’s average throughput, denoted by $T^m(\mathbf{p}^m, \mathbf{p}^{-m})$. The average throughput of every user m depends on the transmission success probability at any

given state i , $\prod_{k \neq m} (1 - p_i^k)$. Hence,

$$T^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \sum_{i=1}^h \pi_i R_i^m p_i^m \prod_{k \neq m} (1 - p_i^k). \quad (3)$$

Each user wishes to optimize a natural tradeoff between throughput and power, which is captured by maximizing the following utility function

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = T^m(\mathbf{p}^m, \mathbf{p}^{-m}) - \lambda^m P^m(\mathbf{p}^m), \quad (4)$$

subject to the power constraint (2), where $\lambda^m \geq 0$ is a user-dependent tradeoff coefficient. We use the notation $\lambda = (\lambda^1, \dots, \lambda^M)$ for the vector of all users' tradeoff coefficients; note that each game *instance* can now be formally described by the tuple $\mathcal{I} = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$.

The term $\lambda^m P^m(\mathbf{p}^m, \mathbf{p}^{-m})$ in (4) can be viewed as the power cost of the mobile. The user utility thus incorporates both a “hard” constraint on power consumption (in the form of (2)), but also accounts for mobile devices that do not consume their power abilities to the maximum extent, as energy might be a scarce resource, the usage of which needs to be evaluated against the throughput benefit. We note that the utility (4) accommodates the following two special cases:

1. *Fully “elastic” users.* By setting $P^m(\mathbf{p}^m) = 1$, a user practically does not have a hard constraint on power usage. Accordingly, the optimal operating point of the user is determined solely by the tradeoff between power and throughput, as manifested by the factor λ^m . The fully elastic user case has been considered in the wireless games literature in different contexts (see, e.g., [1]).
2. *Power-cost neutral users.* Consider a user with $\lambda^m = 0$. Such user is interested only in maximizing its throughput subject to a power constraint. This form of utility has been examined, e.g., in [4] and [8].

2.2.2 Nash Equilibrium

A strategy \mathbf{p}^m of user m is *feasible* if it obeys the power constraint (2). We denote by E^m the feasible strategy space of user m ,

$$E^m = \{\mathbf{p}^m | P^m(\mathbf{p}^m, \mathbf{p}^{-m}) \leq \bar{P}^m, 0 \leq \mathbf{p}^m \leq 1\}; \quad (5)$$

the joint feasible action space is denoted by $E = \prod_{m \in \mathcal{M}} E^m$. A Nash equilibrium point (NE) for our model is a feasible multi-strategy from which no user can unilaterally deviate and improve its utility. Formally, A multi-strategy $\mathbf{p} \triangleq (\mathbf{p}^1, \dots, \mathbf{p}^M)$ is a Nash equilibrium point if

$$\mathbf{p}^m \in \operatorname{argmax}_{\tilde{\mathbf{p}}^m \in E^m} u^m(\tilde{\mathbf{p}}^m, \mathbf{p}^{-m}), \quad \text{for every } m \in \mathcal{M} \quad (6)$$

The existence a Nash equilibrium point is guaranteed, as we summarize below.

Theorem 1. *There always exists a pure Nash equilibrium for the game.*

Proof. Noting that the utility function u^m , (defined in (4)) is linear in the strategy of user m and that the joint feasible region is convex, the existence result follows from a standard use of the Kakutani fixed point theorem (see, e.g., [12]). \square

We conclude this section by a couple of examples which point to some interesting features of the underlying game. The first example shows that there are possibly infinitely many Nash equilibria.

Example 1. Consider a game with two users, m, k , and two states 1, 2. Let $\pi_1 = \pi_2 = \frac{1}{2}$, $R_1^m = R_1^k = 10$, $R_2^m = R_2^k = 5$, $\lambda^m = \lambda^k = 2$, and $\bar{P}^m = 0.8$, $\bar{P}^k = 0.3$. It can be easily shown that the multi-strategy $(p_1^m, p_2^m, p_1^k, p_2^k) = (1, 0.6, 0, x)$ is an equilibrium of the game, for any $x \in [0, 0.6]$.

The next example demonstrates that the behavior of the system in an equilibrium can sometimes be counterintuitive. For example, states which lead to lower expected rates can be utilized (in terms of the total power investment) more than higher quality states.

Example 2. Consider a game with two users, m, k , and two states 1, 2. Let $\pi_1 = \pi_2 = \frac{1}{2}$, $R_1^m = R_1^k = 8$, $R_2^m = R_2^k = 3$, $\lambda^m = \lambda^k = 1$ and $\bar{P}^m = 0.8$, $\bar{P}^k = 0.3$. The unique equilibrium of this game instance is given by $(p_1^m, p_2^m, p_1^k, p_2^k) = (1, 0.6, 0, 0.6)$. Observe that the total power investment at state 1 (0.5) is lower than the total power investment at state 2 (0.6).

Both examples demonstrate some negative indications as to the *predictability* of the Nash equilibrium. Not only the number of equilibria is unbounded, but also we cannot rely on monotonicity results (such as total power investment increasing with the quality of the state) in order to provide a rough characterization of an equilibrium. At the same time, these observations motivate the study of performance-loss bounds at *any* equilibrium point, and also of network dynamics that can converge to a predictable equilibrium point. Both directions would be examined in the sequel.

2.3 Existence of Nash Equilibrium for General Strategy Spaces

In this section we assume that users are not necessarily constrained to stationary strategies and they can utilize nonstationary strategies as well. We show the existence of the Nash equilibrium by showing that equilibria among stationary strategies are actually equilibria among general set of strategies. In other words assuming the system operates at an equilibrium of the stationary strategies then none of the users have incentive to utilize a nonstationary strategy.

The model for nonstationary strategies is slightly different than that of stationary strategies. At each time slot, regardless of the state of the channel and actions of other users, each user has two possible actions. We denote the set of possible actions for user m by $A^m = \{0, 1\}$, where 1 corresponds to transmitting and 0 corresponds to idling. $\mathcal{A} = \{A^m\}_{m=1}^M$ is the joint action space. It is assumed that users may randomize their actions over possible actions at each time slot; we accordingly denote the set of probability distributions over A^m at time slot t by $\Sigma^m(t)$.

We define a general policy of user m by $\mathbf{s}^m = \{s^m(1), s^m(2) \dots\} \in \Sigma^m$ where $s^m(t) \in \Sigma^m(t)$ is a probability distribution through which the user m chooses his action at time slot t , and $\Sigma^m = \Sigma^m(1) \times \Sigma^m(2) \dots$ is the set of probability distributions at all time slots. We denote the policies of all users, or multi-policy of all users, by $\mathbf{s} = \{\mathbf{s}^1, \mathbf{s}^2 \dots \mathbf{s}^M\}$ and policies of users other than m by \mathbf{s}^{-m} .

Policies of users may depend on the past history of the system. The history of user m may include, states observed in the past, actions of user m in the past, collision history, and perhaps some additional information. We denote the history of user m at time t by $y^m(t) \in \mathbf{Y}^m(t)$, where $\mathbf{Y}^m(t)$ is the set of all possible realizations of history of user m up to time t . A general policy of user m at time slot t is a mapping from the history up to time t to possible randomizations over the action space in time slot t , or $s^m(t) : \mathbf{Y}^m(t) \rightarrow \Sigma^m(t)$.

The main problem with nonstationary strategies is the difficulty in determining the feasible action space. Metrics presented in equations (1), (3) are related to expected average power and throughput and determined under the assumption of stationarity in user actions. We next introduce the utility

and constraints used in the nonstationary counterpart of the previously defined problem. To that end, we define the expected average power and expected average utility as,

$$P_{av}^m(\mathbf{s}^m) = \limsup_{T \rightarrow \infty} \frac{1}{T} E\left[\sum_{t=1}^T I^m(t)\right] \quad (7)$$

and

$$u_{av}^m(\mathbf{s}^m, \mathbf{s}^{-m}) = \limsup_{T \rightarrow \infty} \frac{1}{T} E\left[\sum_{t=1}^T I^m(t) \left(\prod_{k \neq m} (1 - I^k(t)) - \lambda^m\right)\right] \quad (8)$$

respectively. Here $I^m(t)$ is an indicator variable that is equal to 1 if user m transmits a packet at time slot t and equal to 0 otherwise. (7) is simply the expected average number of transmissions and (8) follows by noting that the first term is the expected average number of successful transmissions and the second term is the expected average cost of transmissions. Note that although in equations (7),(8), \mathbf{s}^m and \mathbf{s}^{-m} do not appear explicitly, the statistics of indicator variables are determined by these quantities and hence expected average power and utility are functions of \mathbf{s}^m and \mathbf{s}^{-m} .

In the new game formulation the feasible action space of user m will be denoted as follows,

$$E_{av}^m = \{\mathbf{s}^m | P_{av}^m(\mathbf{s}^m) \leq \bar{P}^m, \mathbf{s}^m \in \Sigma^m\}. \quad (9)$$

A scheduling game instance for general strategies can be denoted as $\mathcal{I}_G = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$ similar to our notation for stationary strategies.

Note that each stationary strategy, \mathbf{p} corresponds to a strategy among general strategies that is denoted by $\mathbf{s}(\mathbf{p})$ that satisfies,

$$\Pr(I^m(t) = 1 | H(t) = i) = p_i^m \quad \text{for all } t \in \mathbb{Z}_+. \quad (10)$$

It is easy to see that (using (7) and (8))

$$P_{av}^m(\mathbf{s}^m(\mathbf{p})) = P^m(\mathbf{p}^m) \quad (11)$$

and

$$u_{av}^m(\mathbf{s}^m(\mathbf{p}), \mathbf{s}^{-m}(\mathbf{p})) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}). \quad (12)$$

Hence for a game instance if the stationary strategy \mathbf{p} is feasible the corresponding stationary general strategy $\mathbf{s}(\mathbf{p})$ is also feasible and moreover all players get the same payoff when $\mathcal{I}_G = \mathcal{I}$. Next we state the main theorem of this section. Note that the theorem assumes that the underlying channel state process is Markovian.

Theorem 2. *For a game instance $\mathcal{I} = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$, assume that \mathbf{p} is a NE among stationary strategies. If $\mathcal{I} = \mathcal{I}_G$ and the channel state process is also Markovian then $\mathbf{s}(\mathbf{p})$ is a NE among general strategies.*

Proof. We prove the statement by showing that, at $\mathbf{s}(\mathbf{p})$ no user has an incentive to adopt a non-stationary strategy. In order to simplify the notation we denote the general strategy corresponding to the stationary strategy \mathbf{p} by $\mathbf{s}^m = \mathbf{s}^m(\mathbf{p})$ for any user m .

Assume that the claim is wrong and user m has a strictly better payoff by utilizing a feasible optimal strategy $\hat{\mathbf{s}}^m$ which is not necessarily stationary. Since all users other than m are utilizing stationary strategies and since the channel state process is Markovian it follows that finding $\hat{\mathbf{s}}^m$, maximizing (8) subject to (9) is a constrained Markov decision problem [2, 5]. Moreover the resulting

Markov decision problem has finitely many states (state space is simply \mathcal{H}) and it is known that there exists an optimal stationary solution for this problem [6]. Let \mathbf{q}^m be the described optimal stationary solution, then it follows that

$$u^m(\mathbf{q}^m, \mathbf{p}^{-m}) = u_{av}^m(\hat{\mathbf{s}}^m, \mathbf{s}^{-m}) > u_{av}^m(\mathbf{s}^m, \mathbf{s}^{-m}) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}). \quad (13)$$

and hence \mathbf{p} cannot be an equilibrium as user m has incentive to switch from \mathbf{p}^m to \mathbf{q}^m . Thus we obtain a contradiction and $\mathbf{s}(\mathbf{p})$ is a NE among general strategies as claimed.

Therefore, a pure NE among stationary strategies is a NE among general strategies. \square

In the rest of the paper we restrict ourselves to the stationary game formulation which was previously described.

3 Social Welfare and Threshold Strategies

In this section we characterize the optimal operating point of the network. This characterization allows us to study the efficiency loss due to self-interested behavior (Section 4). Proofs of the theorems in this section can be found in Appendix A.

An optimal multi-strategy in our system is a feasible multi-strategy that maximizes the aggregate user utility. Formally, \mathbf{p}^* is an optimal multi-strategy if it is a solution to the Social Welfare Problem (SWP), given by

$$\begin{aligned} \text{(SWP)} \quad & \max \quad u(\mathbf{p}) \\ & \text{s.t.} \quad \mathbf{p} \in E \end{aligned} \quad (14)$$

where

$$u(\mathbf{p}) = \sum_m T^m(\mathbf{p}) - \lambda^m P^m(\mathbf{p}). \quad (15)$$

We note that (SWP) is a non-convex optimization problem (see Appendix A for a formal proof of this property). Our first result provides a per-state utilization bound for *any* optimal solution of (SWP).

Proposition 1. *Let \mathbf{p} be an optimal solution of (SWP). Then $\sum_m p_i^m \leq 1$ for every $i \in \mathcal{H}$.*

The significance of the above result is that in case that all mobiles use the same power level W for transmission, then the total energy investment is bounded above by Wh , where h is the number of states. Note that this bound does not depend on the number of mobiles. The per-state utilization bound will play a key role in Section 4, while bounding the overall efficiency loss in the system.

For a further characterization of (SWP), we require the next two definitions.

Definition 2 (Partially and Fully Utilized States). *Let \mathbf{p}^m be some strategy of user m . Under that strategy, state i is partially utilized by user m if $p_i^m \in (0, 1)$; state i is fully utilized by the user if $p_i^m = 1$.*

Definition 3 (Threshold Strategies). *A strategy \mathbf{p}^m of user m is a threshold strategy, if the following conditions hold: (i) User m partially utilizes at most one state, and (ii) If user m partially utilizes exactly one state, then the power constraint (2) is active (i.e., met with equality). A multi-strategy $\mathbf{p} = (\mathbf{p}^1, \dots, \mathbf{p}^M)$ is a threshold multi-strategy if \mathbf{p}^m is a threshold strategy for every $m \in \mathcal{M}$.*

The main result of this section is summarized below.

Theorem 3. *There exists an optimal solution of (SWP) where all users employ threshold strategies.*

Due to the non-convexity of (SWP), we cannot rely on first order optimality conditions for the characterization of the optimal solution. Nonetheless, Theorem 3 indicates that there always exist an optimal solution with some well-defined structure, which would be used in the next section for comparing the obtained performance at optimum, to an equilibrium performance.

4 Efficiency Loss

We proceed to examine the extent to which selfish behavior affects system performance. That is, we are interested in comparing the quality of the obtained equilibrium points to the centralized, system-optimal solution (3). Recently, there has been much work in quantifying the efficiency loss incurred by the selfish behavior of users in networked systems (see [13] for a comprehensive review). The two concepts which are most commonly used in this context are the price of anarchy (PoA), which is an upper bound on the performance ratio between the global optimum and the worst Nash equilibrium, and price of stability (PoS), which is an upper bound on the performance ratio between the global optimum and the best Nash equilibrium. The performance measure that we consider here in order to evaluate the quality of a network working point is naturally the aggregate user utility (15).

The standard definitions of PoA and PoS consider all possible instances of the associated game. Recall that in our specific framework, a game instance is given by the tuple $\mathcal{I} = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$. The next example shows that the performance at the *best* Nash equilibrium can be arbitrarily bad compared to the socially optimal working point.

Example 3. Consider a network with two users m and k and two channel states. Let $\pi_1 = \pi_2 = \frac{1}{2}$, $\bar{P}^m = \frac{1}{4}$, $\bar{P}^k = \frac{1}{2}$. Assume that $R_2^m = R_2^k = \epsilon$, $R_1^m = 4$, $R_1^k = 4\epsilon$, $\lambda^m = \lambda^k = \frac{\epsilon}{2}$. The socially optimal working point is given by $\hat{\mathbf{p}} = (\hat{p}_1^m, \hat{p}_2^m, \hat{p}_1^k, \hat{p}_2^k) = (\frac{1}{2}, 0, 0, 1)$ and the unique equilibrium is $\bar{\mathbf{p}} = (\bar{p}_1^m, \bar{p}_2^m, \bar{p}_1^k, \bar{p}_2^k) = (0, \frac{1}{2}, 1, 0)$. Note that $u(\hat{\mathbf{p}}) = 1 + \frac{\epsilon}{8}$, while $u(\bar{\mathbf{p}}) = \frac{3\epsilon}{2}$. Hence, $\frac{u(\hat{\mathbf{p}})}{u(\bar{\mathbf{p}})} > \frac{2}{3\epsilon}$, which goes to infinity as $\epsilon \rightarrow 0$.

The significance of the above example is that if we consider all possible game instances $\{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$, then equilibrium performance can be arbitrarily bad. However, we note that for a *given* mobile technology, some elements within any game instance cannot obtain all possible values. Specifically, π is determined by the technological ability of the mobiles to *quantize* the actual channel quality into a finite number of “information states” as described in section 2. Naturally, one may think of several measures for quantifying the quality of a given quantization. We represent the quantization quality by a single parameter $\pi_{max} \triangleq \max_{i \in \mathcal{H}} \pi_i$, under the understanding that smaller π_{max} , the better is the quantization procedure. In addition, a specific wireless technology is obviously characterized by the power constraint \bar{P}^m . Again, we represent the power-capability of a given technology by a single parameter $P_{min} = \min_{m \in \mathcal{M}} \bar{P}^m$. Finally, we determine the *technological quality* of a set of mobiles through the scalar $Q = \frac{\pi_{max}}{P_{min}}$.

We consider next the efficiency loss for a given technological quality Q . Denote by \mathcal{I}_{Q_0} the subset of all game instances such that $Q = Q_0$.

We define below the price of stability (PoS) and price of anarchy (PoA) as a function of Q .

Definition 4 (Price of Stability - Price of Anarchy). For every game instance \mathcal{I} , denote by $N_{\mathcal{I}}$ the set of Nash equilibria, and let $p_{\mathcal{I}}^*$ be an optimal multi-strategy. Then for any fixed Q , the PoS and

PoA are defined as

$$PoS(Q) = \sup_{\mathcal{I} \in \mathcal{I}_Q} \inf_{\mathbf{p} \in N_{\mathcal{I}}} \frac{u(p_{\mathcal{I}}^*)}{u(\mathbf{p})}, \quad (16)$$

$$PoA(Q) = \sup_{\mathcal{I} \in \mathcal{I}_Q} \sup_{\mathbf{p} \in N_{\mathcal{I}}} \frac{u(p_{\mathcal{I}}^*)}{u(\mathbf{p})}. \quad (17)$$

We next provide upper and lower bounds for $PoS(Q)$ under the assumption that $Q < 1$ (note that an the unbounded price of stability in Example 3 was obtained for $Q > 1$). The upper bound on the price of stability follows from the next proposition.

Proposition 2. *Fix $Q < 1$. Let $\hat{\mathbf{p}}$ be some threshold multi-strategy, and let $u(\hat{\mathbf{p}})$ be the respective aggregate utility (15). Then there exists an equilibrium point $\bar{\mathbf{p}}$ whose aggregate utility is not worse than $u(\hat{\mathbf{p}})(1 - Q)^2$. That is, $\frac{u(\hat{\mathbf{p}})}{u(\bar{\mathbf{p}})} \leq (1 - Q)^{-2}$.*

The key idea behind the proof is to start from a threshold multi-strategy $\hat{\mathbf{p}}$ and to reach an equilibrium point by some iterative process. In each step of the process we obtain the worst-case performance loss, which leads to the overall loss in the entire procedure. A full proof can be found in Appendix B.

Recalling that there always exists an optimal threshold multi-strategy (Theorem 3), immediately establishes the following.

Corollary 1. *Let $Q < 1$. Then $PoS(Q) \leq (1 - Q)^{-2}$.*

The above result implies that for P_{min} fixed, a finer quantization of the channel quality results in a better upper bound for the PoS, which approaches 1 as $\pi_{max} \rightarrow 0$.

It is also possible to obtain a lower bound on the PoS for any given Q as the next proposition suggests.

Proposition 3. *Let $Q < 1$. Then $PoS(Q) \geq \left(1 - \frac{1}{\lfloor \frac{1}{Q} + 1 \rfloor}\right)^{-1}$.*

Proof. We present a parameterized example achieving the PoS lower bound for a given Q . Let Q be fixed and define $j = \lfloor \frac{1}{Q} + 1 \rfloor$. Choose P_{min} such that $P_{min} < \frac{1}{1+j}$ and $\pi_{max} = P_{min}Q$. Let $\mathcal{H} = \{1, 2, \dots, h\}$ and $h > j$. Consider the system with $\pi_i = \pi = \frac{P_{min}}{j} + \epsilon < \pi_{max}$ for sufficiently small ϵ at states $i \in \{1, 2, \dots, j\}$ (i.e. the best j states). Also assume that $\pi_h = \pi_{max}$ and π_i for $j < i < h$ is chosen so that $\sum_{i \in \mathcal{H}} \pi_i = 1$. Let $\bar{P}^m = \bar{P}^k = P_{min}$, $\lambda^m = \lambda^k = \lambda$ and choose rates as $(R_1^m, R_2^m \dots R_h^m) = (10+r_1, 10+r_2 \dots 10+r_j, r_{j+1} \dots r_h)$, $(R_1^k, R_2^k \dots R_h^k) = (\bar{r}_1, \bar{r}_2 \dots \bar{r}_j, \bar{r}_{j+1}, \dots \bar{r}_h)$, for $r_h < \dots < r_{j+1} < \lambda < r_j < \dots < r_1 < \delta$ and $\bar{r}_h < \dots < \bar{r}_{j+1} < \lambda < \bar{r}_j < \dots < \bar{r}_1 < \delta$ for some δ . In this setting optimal solution is $\mathbf{p}^k = 0$, $\mathbf{p}^m = (1, \dots, 1, 1 - \bar{\epsilon}, 0, \dots, 0)$ where state j is the partially used state and $\bar{\epsilon}$ is a function of ϵ that satisfies $\bar{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. On the other hand the best NE satisfies $p_i^m = 1$ for $i < j$ and $p_i^m = 0$ for $j \geq i$ whereas $p_j^k = 1$ and $p_i^k = 0$ for $i \neq j$ (choosing λ such that $\lambda < \bar{\epsilon}r_j$). Now choosing ϵ and δ sufficiently small (so that the contribution of the terms such as r_i and \bar{r}_i to the aggregate utility is negligible) aggregate utility is approximately $10\pi j$ in the central optimum whereas it is $10\pi(j - 1)$ in the best NE, hence

$$PoS(Q) \geq \frac{j}{j-1} = \left(1 - \frac{1}{\lfloor \frac{1}{Q} + 1 \rfloor}\right)^{-1}. \quad (18)$$

as claimed. \square

Observe that, for $Q \ll 1$ or for $Q = \frac{1}{n} + \epsilon$ for some integer n and $0 < \epsilon \ll 1$, $\lfloor \frac{1}{Q} + 1 \rfloor \approx \frac{1}{Q}$ and hence $PoS \geq \frac{1}{1-Q}$ for such Q . Note that $PoS(Q) \leq (1-Q)^{-2}$ by Corollary 1, the gap between the upper and lower bound remains a subject for on-going work.

We conclude this section by showing that the PoA is unbounded for any Q .

Proposition 4. *For any given Q , $PoA(Q) = \infty$*

Proof. The proof is constructive and follows from an example. Fix Q , \mathcal{M} and consider a symmetric game instance with $\mathcal{H} = \{1, 2 \dots h\}$ for $h > Q^{-1}$. Let $R_i^m = R_i$, $\pi_i = \pi_{max} = \frac{1}{h}$, $\lambda^m = \lambda$, $\bar{P}^m = \frac{\pi_{max}}{Q}$ for every $m \in \mathcal{M}$ and $i \in \mathcal{H}$. Assume that $\sum_{i \in \mathcal{H}} \sqrt[M-1]{\frac{\lambda}{R_i}} = h - Q^{-1}$, and $R_i > \lambda$ for every $i \in \mathcal{H}$ (it is always possible to construct such a problem instance for a given Q by choosing h and $\{R_i\}_{i \in \mathcal{H}}$ properly). It can be seen that there exists an equilibrium \mathbf{p} for every such game instance which satisfies

$$p_i^m = 1 - \sqrt[M-1]{\frac{\lambda}{R_i}} \quad \text{for every } m \in \mathcal{M}, i \in \mathcal{H}, \quad (19)$$

which yields $u(\mathbf{p}) = \sum_{m \in \mathcal{M}} u^m(\mathbf{p}) = 0$ at this equilibrium. Note that the given multi-strategy is feasible since for any $m \in \mathcal{M}$,

$$\sum_{i \in \mathcal{H}} \pi_i p_i^m = 1 - \frac{1}{h} \sum_{i \in \mathcal{H}} \sqrt[M-1]{\frac{\lambda}{R_i}} = 1 - \frac{h - Q^{-1}}{h} = \frac{\pi_{max}}{Q} = \bar{P}^m \quad (20)$$

The aggregate utility at a central optimum is obviously greater than 0, as $R_i > \lambda$ for every $i \in \mathcal{H}$, leading to an unbounded PoA. \square

The above result indicates that despite technological enhancements (which result in a low Q), the network can still arrive at bad-quality equilibria with unbounded performance loss. This negative result emphasizes the significance of mechanisms or distributed algorithms, which preclude such equilibria. We address this important design issues in the next section.

5 Best-Response Dynamics

A Nash equilibrium point for our system represents a strategically stable working point, from which no user has incentive to deviate unilaterally. In this section we address the question of if and how the system arrives at an equilibrium, which is of great importance from the system point of view. As discussed in Section 4, the set of equilibria can vary with respect to performance. Hence, we conclude this section by briefly discussing how to lead the system to good quality equilibria.

5.1 Convergence Properties

The most natural mechanism for (distributed) convergence to an equilibrium relies on a user's *best response* (BR), which in general is a user's strategy that maximizes its own utility, given the strategies of other users. In our game, a best response $\bar{\mathbf{p}}^m$ of user m to a given multi-strategy \mathbf{p}^{-m} is given by

$$\bar{\mathbf{p}}^m \in BR^m(\mathbf{p}^{-m}) = \operatorname{argmax}_{\tilde{\mathbf{p}}^m \in E^m} u^m(\tilde{\mathbf{p}}^m, \mathbf{p}^{-m}). \quad (21)$$

An informal description of a general best-response mechanism is simple: *Each user updates its strategy from time to time through a best response, (21).*

The best-response mechanism, described above in its most general form, is not guaranteed to converge to an equilibrium in our system without imposing additional assumptions. We specify below the required assumptions. Our convergence analysis relies on establishing the existence of a *potential* function (see [11]) under a certain condition, which we refer to as the *rate alignment* condition. The rate alignment condition is defined as follows.

Assumption 2 (Rate Alignment Condition). *The set of user rates $\{R_i^m\}_{i \in \mathcal{H}, m \in \mathcal{M}}$ is said to be aligned if there exist per-user positive coefficients $\{c^m\}_{m \in \mathcal{M}}$ and per-state positive constants $\{R_i\}_{i \in \mathcal{H}}$ such that*

$$R_i^m = c^m R_i \quad (22)$$

for every $m \in \mathcal{M}$ and $i \in \mathcal{H}$. The rate alignment condition is satisfied if user rates are aligned.

The coefficient c^m above reflects user m 's relative quality of transmissions, which is affected mainly by its transmission power and location relative to the base station. While the rate alignment condition might not hold for general and heterogeneous mobile systems, a special case of interest which satisfies (22) is the *symmetric-rate* case, i.e., $c^m = c$ for every $m \in \mathcal{M}$. Rate-symmetry is expected in systems where mobiles use the same technology (transmission power and coding scheme), and where "local" conditions, such as distance from the base station, are similar.

For convenience below we give a definition of the ordinal potential games.

Definition 5 (Ordinal Potential Game). *A game is called an ordinal potential game if a function $\phi : E \rightarrow \mathbb{R}$ such that for any $\mathbf{p}^m, \mathbf{q}^m \in E^m$*

$$\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m}) > 0 \Leftrightarrow u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}) > 0 \quad (23)$$

exists, for all m , and for all $\mathbf{p}^{-m} \in \prod_{k \neq m} E^k$. Moreover the function ϕ is called an ordinal potential function.

If (23) is replaced with $\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m}) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})$ then the game is an exact potential game.

Theorem 4. *Under Assumption 2 the collision channel scheduling game is an ordinal potential game with a potential function given by*

$$\phi(\mathbf{p}) = - \sum_{i=1}^h \pi_i R_i \prod_{k \in \mathcal{M}} (1 - p_i^k) - \sum_{i=1}^h \sum_{k \in \mathcal{M}} \pi_i \frac{\lambda^k}{c^k} p_i^k \quad (24)$$

Proof. Consider two different multi-strategies \mathbf{p}, \mathbf{q} such that

$$\begin{aligned} \mathbf{p} &= (\mathbf{p}^m, \mathbf{p}^{-m}) \\ \mathbf{q} &= (\mathbf{q}^m, \mathbf{p}^{-m}) \end{aligned} \quad (25)$$

Observe that

$$\begin{aligned} \phi(\mathbf{p}) - \phi(\mathbf{q}) &= - \sum_i \frac{R_i^m}{c^m} \pi_i ((1 - p_i^m) - (1 - q_i^m)) \prod_{k \neq m} (1 - p_i^k) - \sum_i \frac{\lambda^m}{c^m} \pi_i (p_i^m - q_i^m) \\ &= \frac{1}{c^m} \left(\sum_i \pi_i p_i^m (R_i^m \prod_{k \neq m} (1 - p_i^k) - \lambda^m) - \sum_i \pi_i q_i^m (R_i^m \prod_{k \neq m} (1 - p_i^k) - \lambda^m) \right) \\ &= \frac{1}{c^m} (u^m(\mathbf{p}) - u^m(\mathbf{q})) \end{aligned} \quad (26)$$

Since $c^m > 0$, the above equality implies that the game is an ordinal potential game. \square

The above class of ordinal potential games are also known as *weighted potential games* [11]. The theorem also implies that if $c^m = 1$ for every $m \in \mathcal{M}$, then the game is an exact potential game.

In the following we assume that users restrict themselves to threshold policies (see definition 3). Since our focus is on best response dynamics this assumption is natural as whenever a user updates its policy there always exists a threshold policy that maximizes the performance of that user. Moreover, it turns out that despite the fact that the game we are interested in is an infinite game, the convergence take place in finitely many update periods if users only use threshold policies.

Next lemma shows that our game is a finite game if users are restricted to playing threshold strategies and provides a bound on the number of threshold multi-strategies for any given game instance.

Lemma 1. *For a given game instance with M users and h states the number of threshold multi-strategies is bounded by $(2e)^{M(h+1)}$.*

Proof. Observe that for any user $m \in \mathcal{M}$ threshold policies of m and the extreme points of the feasible region E^m are the same. The idea behind the proof is to upper bound the number of extreme points of the feasible region (for definition of extreme points see Appendix A).

In general a polyhedral region that is a subset of \mathbb{R}^n and is defined by m constraints is represented by

$$\{x \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}, \quad (27)$$

where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$ is a constant vector. Now at any extreme point of (27) at least n linearly independent constraints are active, and such constraints define extreme points uniquely, hence there are at most, $\binom{m}{n}$ threshold policies.

In the scheduling problem discussed in this paper, each user has h decision variables and a total of $2h + 1$ constraints. Hence for our particular problem there are $M(2h + 1)$ constraints and Mh variables in total. Thus the number of threshold policies is bounded by

$$\binom{M(2h+1)}{Mh} = \binom{M(2h+1)}{M(h+1)} \leq \left(\frac{eM(2h+1)}{M(h+1)}\right)^{M(h+1)} \leq (2e)^{M(h+1)}, \quad (28)$$

where the first inequality follows from the inequality $\binom{m}{n} \leq \left(\frac{em}{n}\right)^n$. \square

Throughout this section, we assume that users may update their strategy at a slower time-scale compared to their transmission rates. For simplicity, we assume that user updates may take place only every T_E time slots and refer to T_E as the *update period*.

For our convergence result, we require the following set of assumptions.

Assumption 3.

- (i) *The user population is fixed.*
- (ii) *Rates are aligned (see Definition 2).*
- (iii) *The transmission-success probabilities $\prod_{k \neq m} (1 - p_i^k)$, $i \in \mathcal{H}$ are perfectly estimated by each user before each update.*

Consider the following mechanism.

Definition 6 (Round-Robin BR Dynamics). *Strategy updates take place in a round-robin manner and at each update period only a single user may modify its strategy. The user who is chosen for*

update modifies its strategy to a threshold strategy from the set $BR^m(\mathbf{p}^{-m})$, if the modification strictly improves its utility.

Note that due to linearity of the best response problem for all users, there always exists a threshold strategy in $BR^m(\mathbf{p}^{-m})$.

The basic convergence property is stated below.

Theorem 5. *Let Assumption 3 hold. Then best response dynamics converge in finitely many update periods to an equilibrium point. In addition, the number of update periods required for convergence is upper bounded by $M(2e)^{M(h+1)}$.*

Proof. By restricting users to threshold strategies, the underlying game becomes a finite game (i.e., the game has a finite action space as Lemma 1 suggests), with a potential function given by (24). As such, the finite improvement property (FIP) in potential games holds, that is any sequence of updates which results in strict improvement in the utility of the user who is modifying its strategy is finite (see [11]). Moreover, each finite improvement path terminates in a NE.

By Lemma 1 the number of threshold strategies is bounded by $(2e)^{M(h+1)}$, and observing that no multi-strategy can occur more than once during the updates (as the potential strictly increases with each update) this implies that number of updates (in which a user modifies its strategy) required for convergence is bounded by $(2e)^{M(h+1)}$. By Definition 6 before convergence, a user who has an estimation of transmission-success probabilities and incentive to modify its strategy is chosen by the round robin schedule at most in M updates and hence number of updates required for convergence is bounded by $M(2e)^{M(h+1)}$. \square

We emphasize that the restriction to threshold strategies is commensurate with the users' best interest. Not only there always exists such best-response strategy, but also it is reasonably easier to implement.

We discuss next some important considerations regarding the presented mechanism and the assumptions required for its convergence. The best response dynamics as described in definition 6, requires synchronization between the mobiles, which can be done centrally by the base station or by a supplementary distributed procedure. We emphasize that the schedule of updates is the only item that needs to be centrally determined. Users are free to choose their strategies according to their own preferences, which are usually private information. Assumption 3(iii) entails the notion of a quasi-static system, in which each user responds to the steady state reached after preceding user update. This approximates a natural scenario where users update their transmission probabilities at a much slower time-scales than their respective transmission rates. An implicit assumption here is that the update-period T_E is chosen large enough to allow for accurate estimation of the transmission-success probabilities. We leave the exact determination of T_E for future work. We emphasize that users need not be aware of the specific transmission probabilities p_i^m of other users. Indeed, in view of (4), only the transmission-success probabilities $\prod_{k \neq m} (1 - p_i^k)$, $i \in \mathcal{H}$ are required. These can be estimated by sensing the channel and keeping track of idle slots.

A last comment relates to the rate-alignment condition. The convergence results in this section rely on establishing a potential function for the underlying game, which is shown to exist when rates are aligned. In section 6, we show that in a system of three states or more, the alignment condition is not only sufficient, but also necessary for the existence of a potential. This suggests that novel methods would have to be employed for establishing convergence of dynamics under more general assumptions.

Next we relax the deterministic update schedule (round-robin updates) of the previous theorem. Consider the following update rule,

Definition 7 (Randomized BR Dynamics). Let $f_{\mathcal{M}} : \mathcal{M} \rightarrow [0, 1]$ be a probability mass function defined on set \mathcal{M} such that $f_{\mathcal{M}}(k) > 0$ for all $k \in \mathcal{M}$. Start from a multi-strategy \mathbf{p} . At each update period,

1. Randomly choose one user in \mathcal{M} according to $f_{\mathcal{M}}$.
2. Let m be the user chosen in the previous step, set $\hat{\mathbf{p}}^m$ to a threshold best response of user m in $BR^m(\mathbf{p}^{-m})$.
3. If $u^m(\hat{\mathbf{p}}^m, \mathbf{p}^{-m}) > u^m(\mathbf{p}^m, \mathbf{p}^{-m})$ then let $\mathbf{p} = (\hat{\mathbf{p}}^m, \mathbf{p}^{-m})$, otherwise do not modify \mathbf{p} .

Theorem 6. Let Assumption 3 hold. Then the randomized best response dynamics converge to a NE of the system in finitely many updates with probability 1.

Proof. As in Theorem 5 game is a finite ordinal potential game, and has the finite improvement property.

Let K be the length of the longest improvement path, since the game is a finite game there are finitely many improvement paths and K is well defined. Using the randomized best response dynamics at each update period assuming that a NE is not reached, with probability at least $\min_{k \in \mathcal{M}} f_{\mathcal{M}}(k) > 0$ a user who has incentive to modify his policy is chosen for update. The expected number of updates to reach to a NE (N_{NE}) is smaller than the expected time to observe K successes in a Bernoulli process (T_K) with success probability $\min_{k \in \mathcal{M}} f_{\mathcal{M}}(k)$. The latter is simply $\frac{K}{\min_{k \in \mathcal{M}} f_{\mathcal{M}}(k)}$, hence

$$E[N_{NE}] \leq E[T_K] = \frac{K}{\min_{k \in \mathcal{M}} f_{\mathcal{M}}(k)}. \quad (29)$$

Thus, with probability 1 a NE is achieved in finitely many updates. Now the result follows since when a NE is achieved, none of the users have any incentive to deviate from the NE. \square

This result also implies that if estimations are available in T_E amount of time then expected convergence time to a NE is bounded by $\frac{KT_E}{\min_{k \in \mathcal{M}} f_{\mathcal{M}}(k)}$ time slots.

5.2 Experiments

The objective in this subsection is to study through simulations the convergence properties of sequential best-response dynamics. More specifically, we wish to examine the dependence of convergence time on several factors, such as the number of users in the system, the number of states, and the technology factor Q . In all our experiments, we consider a relaxed version of Assumption 3, where the rate-alignment condition (Assumption 3(i)) is not enforced.

The specific setup for our simulations is as follows. We assume that $\pi_i = \frac{1}{h}$ for every $i \in \mathcal{H}$. For given Q , M and h , we construct a significant number of game instances (10000) by randomly choosing in each instance the power constraints \bar{P}^m , the tradeoff coefficient λ^m and the associated rates R_i^m for every $m \in \mathcal{M}$, $i \in \mathcal{H}$. We simulate each game instance, and examine the average convergence speed, measured in the number of round-robin iterations (i.e., each user updates its strategy exactly once in every iteration, Round-Robin BR dynamics are utilized). Figure 2 presents the convergence speed results for two different values of Q , as a function of the number of users in the system. For each value of Q , we consider three different number of states h .

As seen from Figure 2, the average number of iterations required for convergence is less than three on average. We emphasize that all game instances converge without requiring the rate-alignment condition, indicating the possibility to exclude this condition in future analysis of best-response convergence. We observe that all graph curves initially increase as a function of the number of users,

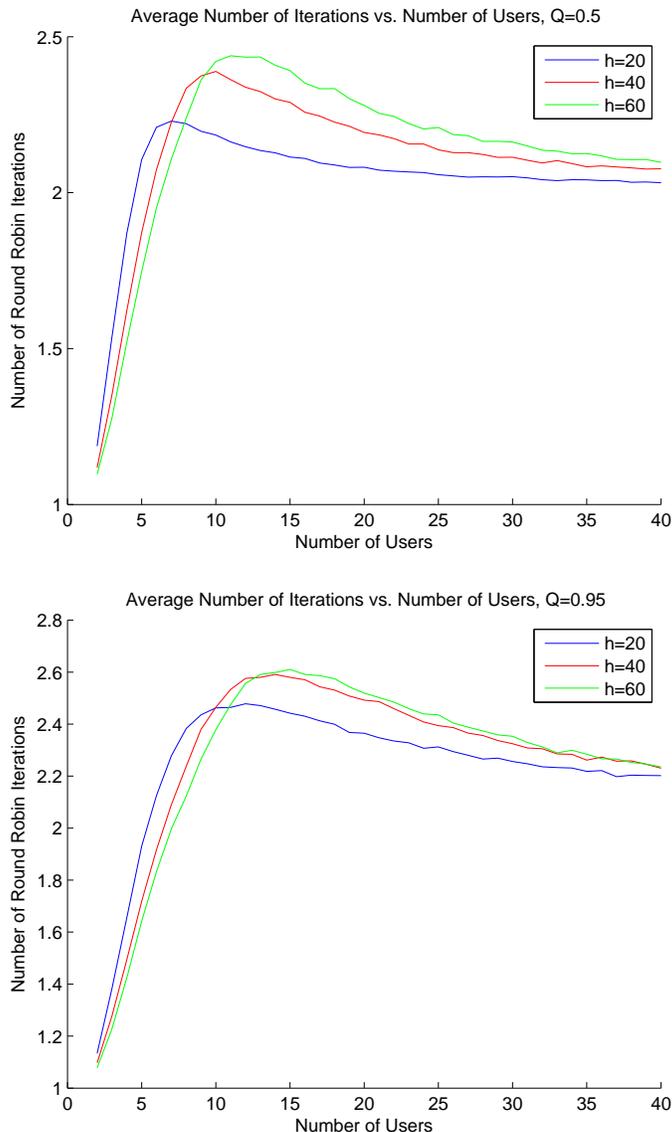


Figure 2: Convergence speed as a function of the number of users.

and at some point gradually decrease until reaching a fixed number of iterations. This interesting phenomenon can be intuitively justified as follows: When the number of users is relatively small, there is less competition on each state, and convergence is fast. At the other extreme, when the number of users is larger than some threshold, then there are more users who can fully utilize states at the first iteration (see Definition 2), thereby decreasing the competition at subsequent iterations and leading to faster convergence.

5.3 Obtaining Desirable Equilibria

We conclude this section by briefly discussing possible means for obtaining high-quality equilibria in terms of the aggregate utility (15). Theorem 5 introduces a scheme (or mechanism) which assures converge to an equilibrium point in a finite number of steps. However, the resulting equilibrium can be of low quality. Proposition 2 suggests that if the system is initiated at some threshold multi-strategy, then the equilibrium performance cannot deviate by much, compared to the performance

at the initial working point. Consequently, one may consider an iterative *hybrid* algorithm, in which a network-management entity forces some initial working-point (a threshold multi-strategy), waits enough time until convergence, and if the equilibrium performance is unsatisfactory, enforces a different working point, until reaching a satisfactory equilibrium. Such algorithm relies on the fast convergence to an equilibrium, which is demonstrated in all our simulations, and allows to consider numerous initial working points in plausible time-intervals.

Obviously, such algorithm requires some central knowledge at the management entity, such as the different user preferences (or at least estimates thereof). The precise requirements and properties of such an algorithm, as well as the means for enforcing initial working-points, remain as a challenging future direction.

6 Condition for the Existence of Differentiable Ordinal Potential

In this section we show that our scheduling game is not an ordinal potential game if assumption 2 is not satisfied. To this end, we obtain necessary conditions for the existence of a differentiable ordinal potential for general n-person games. Consequently we show that the scheduling game is an ordinal potential game if and only if assumption 2 is satisfied.

Some of the results in this section are not limited to particular scheduling game. For this reason we denote by $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m(\cdot)\} \rangle$ a generic game with user set \mathcal{M} , utility functions $u^m(\cdot)$ and feasible action space E^m for every user $m \in \mathcal{M}$. We assume that $E^m \subset \mathbb{R}^h$ for some $h \in \mathbb{Z}_+$ and use \mathbf{p}^m to denote the actions of user m . The multi-strategy \mathbf{p} denotes strategies of all users and $E = \prod_{m \in \mathcal{M}} E^m$. We also assume that $u^m(\cdot)$ is differentiable in \mathbf{p}^m .

In the following proposition, we study the relationship between the partial derivatives of ordinal potential function and the utilities of users. To this end we are interested in the set $A^m = \{\mathbf{p} \mid \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \neq 0, \text{ for some } i \in \mathcal{H} \text{ and } \mathbf{p} \in \text{int}(E)\}$ ($\text{int}(\cdot)$ denotes interior of a set) for all $m \in \mathcal{M}$. The proposition states that in ordinal potential games for any user $m \in \mathcal{M}$, at any multi-strategy $\mathbf{p} \in A^m$, the vector of partial derivatives of the ordinal potential function and that of the utility of user m with respect to the actions of user m are aligned with some alignment function $d^m(\mathbf{p}^m, \mathbf{p}^{-m}) : E \rightarrow \mathbb{R}$. We start with the following proposition,

Proposition 5. Consider the game $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m(\cdot)\} \rangle$,

(i) If a continuously differentiable ordinal potential function $\phi(\cdot) : E \rightarrow \mathbb{R}$ exists then for every $m \in \mathcal{M}$ it satisfies,

$$\frac{\partial \phi(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \quad \text{for all } \mathbf{p} \in A^m, i \in \mathcal{H} \quad (30)$$

where $d^m(\mathbf{p}) \geq 0$ for all $\mathbf{p} \in A^m$

(ii) If for all $m \in \mathcal{M}$, utility functions $u^m(\cdot)$ are linear in \mathbf{p}^m , and if there exists a continuously differentiable function $\phi(\cdot) : E \rightarrow \mathbb{R}$ such that for all $m \in \mathcal{M}$,

$$\frac{\partial \phi(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \quad \text{for all } \mathbf{p} \in E, i \in \mathcal{H} \quad (31)$$

where $d^m(\mathbf{p}) > 0$ for all $\mathbf{p} \in E$ then $\phi(\cdot)$ is an ordinal potential function for \mathcal{G} .

Proof. (i) Observe first that (30) implies that for user m the vector of partial derivatives of his utility

and the ordinal potential function with respect to \mathbf{p}^m are aligned. Assume that a potential function $\phi(\cdot) : E \rightarrow \mathbb{R}$ exists, and assume by contradiction that (30) does not hold for some m . Then there exists $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m})$ and $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m})$, $\mathbf{p} \in A^m$, $\mathbf{q} \in E$ such that

$$\nabla u^m(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0 \quad \text{and} \quad \nabla \phi(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) < 0 \quad (32)$$

This implies that the directional derivatives of $u^m(\cdot)$ and $\phi(\cdot)$ in $(\mathbf{q} - \mathbf{p})$ direction have opposite signs at \mathbf{p} , and hence there exists some $\epsilon > 0$ small such that

$$u^m(\mathbf{p}^m + \epsilon(\mathbf{q}^m - \mathbf{p}^m), \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0 \quad (33)$$

whereas

$$\phi(\mathbf{p}^m + \epsilon(\mathbf{q}^m - \mathbf{p}^m), \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) < 0 \quad (34)$$

which is a contradiction to the assumption that $\phi(\cdot)$ is an ordinal potential function.

(ii) Assume that (31) holds for some function $\phi(\cdot)$. Then, for every $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m}) \in E$ and $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m}) \in E$

$$\nabla u^m(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0 \Leftrightarrow \nabla \phi(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0 \quad (35)$$

Observe that since utility of user m is linear in his actions,

$$u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \nabla u^m(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) \quad (36)$$

Moreover, for all $m \in \mathcal{M}$ linearity also implies that

$$\frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = \frac{\partial u^m(\gamma^m, \mathbf{p}^{-m})}{\partial p_i^m} \quad \text{for all } \gamma^m \in E^m, i \in \mathcal{H} \quad (37)$$

Hence, substituting (37) in (36) yields,

$$u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) \quad (38)$$

for any $\gamma^m \in E^m$.

First we show that $u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0 \Rightarrow \phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$. If $u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$ then by (38) $\nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0$. For $\gamma^m = \alpha \mathbf{q}^m + (1 - \alpha) \mathbf{p}^m$, $\alpha \in (0, 1)$ this is equivalent to $\nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - (\gamma^m, \mathbf{p}^{-m})) > 0$, and using (35), the last inequality implies that $\nabla \phi(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - (\gamma^m, \mathbf{p}^{-m})) > 0$. Now using the fundamental theorem of calculus,

$$\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) = \int_{\Gamma^m} \nabla \phi(\mathbf{s})^T d\mathbf{s} > 0 \quad (39)$$

where $\Gamma^m = \{(\alpha \mathbf{q}^m + (1 - \alpha) \mathbf{p}^m, \mathbf{p}^{-m}) \mid \alpha \in [0, 1]\}$. In (39) we made use of the fact that in Γ^m , \mathbf{s} is in the form of $(\gamma^m, \mathbf{p}^{-m})$, hence vectors $\mathbf{q} - (\gamma^m, \mathbf{p}^{-m})$ and $d\mathbf{s}$ are always aligned and $\nabla \phi(\cdot)$ is a continuous function.

Next we show that $\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0 \Rightarrow u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$. If $\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$ then there exists a $\gamma^m = \alpha \mathbf{q}^m + (1 - \alpha) \mathbf{p}^m$ for some $\alpha \in [0, 1]$ so that $\nabla \phi(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0$ since otherwise we obtain a contradiction with $\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$ using the integral in (39). Combining this with (35) it can be obtained that $\nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0$. Hence, (37) and (36) imply that $u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$.

Therefore, if (31) is satisfied with some continuously differentiable $\phi(\cdot)$, the game \mathcal{G} is an ordinal potential game with potential function $\phi(\cdot)$. \square

Using proposition 5 we show below that our game does not possess a potential function unless assumption 2 holds. To this end we first state the following result,

Lemma 2. *In the scheduling game, the set $B = \{\mathbf{p} \mid \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \neq 0, \text{ for all } m, i \text{ and } \mathbf{p} \in \text{int}(E)\}$ contains a nonempty open set of the joint feasible action space, E .*

Proof. For every nonempty open subset U of E and for a user k and a state j , there exists an open set V , contained in U such that for every multi-strategy \mathbf{q} in V , $\frac{\partial u^k}{\partial p_j^k}(\mathbf{q}) \neq 0$, since $\frac{\partial u^k}{\partial p_j^k}(\mathbf{q})$ is a continuous function of its argument and the set $\mathbb{R} - \{0\}$ is open. The fact that V is not empty immediately follows from the definition of the utility function $u^k(\cdot)$. Since the above statement is true for an arbitrary open set U and since there are finitely many users and states in the system, there exists a nonempty open neighborhood of E which is contained in B . \square

In the following we denote the nonempty open set of E contained in B by V_0 .

The next lemma characterizes existence of the C^2 ordinal potential function, i.e. twice differentiable ordinal potential function with continuous partial derivatives, for the scheduling game.

Lemma 3. *Consider the scheduling game with C^2 ordinal potential function with $|\mathcal{M}| > 1$, $|\mathcal{H}| > 2$. Let ϕ , and alignment functions, $d^m(\cdot)$, $d^k(\cdot)$ be as in (i) of proposition 5. For any $m, k \in \mathcal{M}$, there exists $\alpha^{mk} : E \rightarrow \mathbb{R}$ such that for every $\mathbf{p} \in B$ and for any $i \in \mathcal{H}$,*

$$\frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^k} = \alpha^{mk}(\mathbf{p}) \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^k} \quad \text{for all } i \in \mathcal{H} \quad (40)$$

and

$$\frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m} = \alpha^{mk}(\mathbf{p}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \quad \text{for all } i \in \mathcal{H}. \quad (41)$$

Proof. Consider two different users m, k and two different states i, j . Then by Proposition 5 and by the symmetry of the second derivatives it follows that

$$\frac{\partial^2 \phi}{\partial p_j^k \partial p_i^m}(\mathbf{p}) = \frac{\partial}{\partial p_j^k} (d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m}) \quad (42)$$

$$= \frac{\partial}{\partial p_i^m} (d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_j^k}) = \frac{\partial^2 \phi}{\partial p_i^m \partial p_j^k}(\mathbf{p}) \quad (43)$$

for $\mathbf{p} \in B \subset A$ defined as in lemma 2.

The previous equation is equivalent to,

$$\frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_j^k} \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = \frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m} \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_j^k} \quad (44)$$

using chain rule and observing that partial derivative of a utility of a user with respect to actions in some state j , is a function of actions of users in state j . Now we choose,

$$\alpha^{mk}(\mathbf{p}) = \frac{\frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_j^k}}{\frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_j^k}} = \frac{\frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m}}{\frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m}}. \quad (45)$$

Note that if there are more than two states, by fixing i and modifying j or fixing any i and modifying j and using (44), (45) the result follows. \square

The underlying reason for (40) and (41) to hold is that for fixed \mathbf{p} the system of equations in (44) with unknowns equal to partial derivatives of d^m and d^k is a linear system of equations with null space of rank one, and null space vector satisfies (40) and (41). However, if there are two or less states in the system, this system of equations has a null space with a higher dimension and hence (40) and (41) does not follow.

The next theorem suggests that a C^2 ordinal potential function, does not exist in the game unless the rate alignment condition holds. A natural question to ask is whether an ordinal potential function which is not twice differentiable can exist, and the answer is no. However, the proof is detailed and will be included in a later paper.

Theorem 7. *Consider a scheduling game with more than a single player and three or more states. The game has a C^2 ordinal potential function if and only if assumption 2 holds.*

Proof. If assumption 2 holds, the result follows directly from Theorem 4.

For the other part of the claim, assume that there exists a C^2 potential function ϕ for the scheduling game.

Observe that there exists $\mathbf{p} \in V_0$ such that $d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \neq 0$ or $d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \neq 0$ since otherwise there exists a neighborhood in which although utility of a user is changing by modifying the policy the potential of the game remains constant. Fix such a $\mathbf{p} \in V_0$ such that $d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \neq 0$.

Now using symmetry of derivatives for p_i^m and p_i^k one gets,

$$\begin{aligned} \frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^k} \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} + d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial^2 u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^k \partial p_i^m} = \\ \frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m} \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^k} + d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \frac{\partial^2 u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m \partial p_i^k} \end{aligned} \quad (46)$$

Using (40) and (41) one can see that terms including partial derivatives of d^m and d^k cancel, and substituting the second partial derivatives of utilities one can achieve,

$$R_i^m d^m(\mathbf{p}^m, \mathbf{p}^{-m}) = d^k(\mathbf{p}^k, \mathbf{p}^{-k}) R_i^k \quad (47)$$

Since $d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \neq 0$ it follows that $d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \neq 0$.

Note that (47) holds for any i . Therefore, (47) implies that rate alignment condition holds, hence the scheduling game is an ordinal potential game if and only if assumption 2 holds. \square

7 Conclusion

We have considered in this paper a wireless network game, where mobiles interact over a shared collision channel. The novelty in our model is the state correlation assumption, which incorporates the effect of global time-varying conditions on performance. In general, the correlated state can be exploited by the users for time-division of their transmission, which would obviously increase the system capacity. However, we have shown that under self-interested user behavior, the equilibrium performance can be arbitrarily bad. Nevertheless, the efficiency loss at the best equilibrium can be bounded as a function of a technology parameter, which accounts both for the mobiles power limitations and the level of discretization of the underlying channel quality. Importantly, we have shown that best-response dynamics may converge to an equilibrium in finite time, and empirically verified that such dynamics converge fairly fast.

We briefly note several extensions and open directions: The convergence analysis of best-response dynamics under more general conditions is of great interest. As mentioned, new tools rather than the use of a potential function seem to be necessary. Another challenging direction is to obtain a tight bound on the price of stability, and examine how the price of anarchy can be bounded while fixing other game parameters besides the technological quality. At a higher level, one may consider the *partial* correlation case, in which a user reacts to a channel state that incorporates both global and local temporal conditions.

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APPENDICES

A Proofs for Section 3

First we prove that the (SWP) is a nonconvex optimization problem. For this we make use of the Hessian matrix of $u(\mathbf{p})$, denoted by $\nabla^2 u(\mathbf{p})$. The entries of the Hessian for a function $f(\mathbf{x})$ can be given as $\nabla^2 f(\mathbf{x})_{i,j} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x})$.

Lemma 4. *(SWP) is a nonconvex optimization problem.*

Proof. The definition of $u(\mathbf{p})$ in (15) reveals that the diagonal of the $\nabla^2 u(\mathbf{p})$ is always equal to zero. Hence, trace of the Hessian, or equivalently the sum of eigenvalues of the Hessian is equal to zero for any \mathbf{p} . But for $\mathbf{p} \in E$ such that $p_i^m \in (0, 1)$ for all m, i , the Hessian is not identically equal to zero, or equivalently all eigenvalues of it can not be equal to zero. Thus, the Hessian is neither negative nor positive semidefinite. Since Hessian is not negative semidefinite, $u(\mathbf{p})$ is not a concave function of its argument and hence the (SWP) is not a convex optimization problem. \square

Proof of Proposition 1. For the proof, we shall make use of the partial derivatives of the aggregate utility, given by

$$\frac{\partial u(\mathbf{p})}{\partial p_i^m} = \pi_i (R_i^m \prod_{k \neq m} (1 - p_i^k) - \lambda_i^m - \sum_{l \neq m} p_i^l R_i^l \prod_{k \neq m, l} (1 - p_i^k)). \quad (48)$$

Let $\rho_i = \prod_k (1 - p_i^k)$, for any $\mathbf{p} \in \text{int}(E)$ (48) can be rewritten as

$$\frac{\partial u(\mathbf{p})}{\partial p_i^m} = \pi_i \frac{\rho_i}{1 - p_i^m} \left(R_i^m - \frac{\lambda_i^m}{\rho_i} (1 - p_i^m) - \sum_{l \neq m} \frac{p_i^l}{1 - p_i^l} R_i^l \right) \quad (49)$$

$$= \pi_i \frac{\rho_i}{1 - p_i^m} \left(\frac{R_i^m}{1 - p_i^m} - \frac{\lambda_i^m}{\rho_i} (1 - p_i^m) - \sum_l \frac{p_i^l}{1 - p_i^l} R_i^l \right). \quad (50)$$

where we used the fact that $1 - p_i^m > 0$ and $\rho_i > 0$ in the interior of the feasible region.

Let \mathbf{p} be an optimal solution of (SWP) and consider state i . If this state is used by a user with probability 1 then obviously no other user transmits at this state and the claim immediately follows.

The claim obviously holds if no user utilizes the state. Hence, assume that state i is partially used by some users. Let $K_i \subset \mathcal{M}$ be the subset of users that partially utilize state i . Let,

$$m \in \operatorname{argmin}_{l \in K_i} \frac{R_i^l}{1 - p_i^l}. \quad (51)$$

Since \mathbf{p} is optimal it follows that

$$\frac{\partial u(\mathbf{p})}{\partial p_i^m} \geq 0 \quad (52)$$

since otherwise the aggregate utility can be improved by decreasing p_i^m . Substituting (50) in (52) and recalling that $1 - p_i^k > 0$ for every $k \in K_i$

$$\frac{R_i^m}{1 - p_i^m} - \frac{\lambda_i^m}{\rho_i} (1 - p_i^m) - \sum_l \frac{p_i^l}{1 - p_i^l} R_i^l \geq 0 \quad (53)$$

and hence

$$\frac{R_i^m}{1-p_i^m} \geq \sum_l \frac{p_i^l}{1-p_i^l} R_i^l \geq \frac{R_i^m}{1-p_i^m} \sum_l p_i^l \quad (54)$$

where the last inequality follows from (51). (54) immediately implies that $\sum_l p_i^l \leq 1$. \square

Below we show that an optimal solution of the (SWP) is a threshold multi-strategy. In the proof we study some linear optimization problems related to (SWP). In linear programming the *boundary* of the feasible region is defined as the set of operating points in which at least one constraint is active (satisfied with equality) and an *extreme point* of the feasible region is an operating point in which there are n linearly independent active constraints, if the number of unknowns in the problem is n . Observe that for any user $m \in \mathcal{M}$ threshold policies of m and the extreme points of the feasible region E^m are the same. By studying the Hessian matrix it can be shown that the optimal solution of the (SWP) is always at the boundary of the feasible region E . Existence of threshold optimal solutions is proved below.

Proof of Theorem 3. Let $\hat{\mathbf{p}}$ be an optimal solution of the (SWP), define a function $g_{\hat{\mathbf{p}}}^m : E^m \rightarrow \mathbb{R}$, such that

$$g_{\hat{\mathbf{p}}}^m(\mathbf{p}^m) \triangleq u(\mathbf{p}^m, \hat{\mathbf{p}}^{-m}) - u(\hat{\mathbf{p}}^m, \hat{\mathbf{p}}^{-m}). \quad (55)$$

The function $g_{\hat{\mathbf{p}}}^m(\cdot)$ quantifies the change in the aggregate utility, if user m uses a strategy \mathbf{p}^m instead of $\hat{\mathbf{p}}^m$. Consider the following optimization problem,

$$\begin{aligned} \max \quad & g_{\hat{\mathbf{p}}}^m(\mathbf{p}^m) \\ \text{s.t.} \quad & \mathbf{p}^m \in E^m. \end{aligned} \quad (56)$$

If an optimal solution of this maximization problem is \mathbf{p}^m it follows from definition of $g_{\hat{\mathbf{p}}}^m$ that $(\mathbf{p}^m, \hat{\mathbf{p}}^{-m})$ is an optimal solution of the (SWP).

Observe that $g_{\hat{\mathbf{p}}}^m(\mathbf{p}^m)$ is linear in \mathbf{p}^m . Since E^m is by definition a polyhedron (see (5)), (56) is a linear optimization problem. Therefore, an optimal solution of (56) exists at an extreme point of E^m , and it follows that there exists an optimal solution of the (SWP) in which user m uses a threshold strategy.

Note that in the above argument starting from an arbitrary optimal solution of (SWP), an optimal solution of the (SWP) in which all users but m utilize the same strategy and user m utilizes a threshold strategy is achieved. Repeating the same argument for all users it follows that there exists an optimal solution of the (SWP) where all users use threshold policies. \square

B Proof for Proposition 2

Below we construct a NE starting from the initial multi-strategy $\hat{\mathbf{p}}$.

1. Let $\hat{\mathcal{H}}$ be the set of states such that each state i belonging to $\hat{\mathcal{H}}$ satisfies $\hat{p}_i^m > 0$ for some $m \in \mathcal{M}$. For each $i \in \hat{\mathcal{H}}$, define

$$m_i \in \operatorname{argmax}_{\{k \in \mathcal{M} | 0 < p_i^k \leq 1\}} R_i^k - \lambda^k \quad (57)$$

If the set $\operatorname{argmax}_{\{k \in \mathcal{M} | 0 < p_i^k \leq 1\}} R_i^k - \lambda^k$ is not a singleton, m_i is chosen arbitrarily from the elements of the set. Consider a modified multi-strategy \mathbf{q} of the original multi-strategy $\hat{\mathbf{p}}$ given

by,

$$q_i^k = \begin{cases} 1 & \text{if } i \in \mathcal{H}, k = m_i \\ 0 & \text{otherwise} \end{cases} \quad (58)$$

Let $N = \{m_i | i \in \mathcal{H}\}$. Note that from the definition of the threshold policy and (57) it follows that the transmission probability of any user in \mathbf{q} is strictly larger than the transmission probability in $\hat{\mathbf{p}}$ at most in a single state (namely the partially used state if it exists).

2. The multi-strategy \mathbf{q} can be infeasible, as the power constraint of every $m \in N$ can be violated by investing extra power in partially used states. Note that if $k \notin N$, strategy $\mathbf{q}^k \leq \hat{\mathbf{p}}^k$ is feasible. Also $q_i^k \in \{0, 1\}$ for all $k \in \mathcal{M}$, $i \in \mathcal{H}$, and no two users utilize the same state.

Let $\Delta \bar{P}^m$, $m \in N$ denote the maximum additional power investment required for user m to utilize strategy \mathbf{q}^m instead of $\hat{\mathbf{p}}^m$ (recall that each user partially uses at most a single state). This quantity is obviously bounded by π_{max} , since fully utilizing any state requires at most π_{max} amount of additional power. Set $\Delta \bar{P}^m = 0$ if the multi-strategy of m is already feasible. We next obtain a feasible strategy by modifying \mathbf{q} .

3. Consider the following optimization problem,

$$\begin{aligned} \overline{BR}^k(\tilde{\mathbf{p}}^k, \mathbf{p}^{-k}) &= \underset{\mathbf{p}^k}{\operatorname{argmax}} \quad u^k(\mathbf{p}^k, \mathbf{p}^{-k}) \\ \text{s.t.} \quad &\sum_{i \in \mathcal{H}} \pi_i p_i^k \leq \sum_{i \in \mathcal{H}} \pi_i \tilde{p}_i^k \\ &0 \leq p_i^k \leq 1. \end{aligned} \quad (59)$$

$\overline{BR}^k(\tilde{\mathbf{p}}^k, \mathbf{p}^{-k})$ denotes the threshold best response of user k to \mathbf{p}^{-k} assuming that the power constraint in the optimization problem is equal to power investment of $\tilde{\mathbf{p}}^k$. Due to linearity of $u^k(\mathbf{p}^k, \mathbf{p}^{-k})$ in \mathbf{p}^k , the problem becomes a linear optimization problem and a threshold strategy solving (59) always exists.

Define,

$$\gamma_k = \max \{j \in \mathcal{H} \cup \{h+1\} | \sum_{i=j}^h \pi_i q_i^k \geq \Delta \bar{P}^k\}. \quad (60)$$

Here assume that $\sum_{i=h+1}^h \alpha_i = 0$ for any α_i by convention, thus $\gamma_k = h+1$ for any feasible strategy \mathbf{q}^k .

Consider the following iterative algorithm

- (a) Set $\bar{M} = \emptyset$, $\mathbf{p} = \mathbf{q}$.
- (b) Choose $k \in \operatorname{argmin}_{l \in \mathcal{M} - \bar{M}} \gamma_l$ (if the set is not a singleton choose an arbitrary k in the set).
- (c) If $\gamma_k < h+1$ modify \mathbf{q}^k to

$$\bar{q}_i^k = \begin{cases} q_i^k & \text{if } i < \gamma_k \\ \frac{\sum_{l=1}^{\gamma_k} \pi_l q_l^k - \Delta \bar{P}^k}{\pi_{\gamma_k}} & \text{if } i = \gamma_k \\ 0 & \text{if } i > \gamma_k \end{cases} \quad (61)$$

else $\bar{\mathbf{q}}^k = \mathbf{q}^k$. Let $\mathbf{w}^k = \overline{BR}^k(\bar{\mathbf{q}}^k, \mathbf{p}^{-k})$.

- (d) For any $i \in \mathcal{H}$ if $0 < w_i^k < 1$ set $w_i^k = 0$.

(e) Set $\mathbf{p} = (\mathbf{w}^k, \mathbf{p}^{-k})$, $\bar{M} = \bar{M} \cup \{k\}$. If $\bar{M} = \mathcal{M}$ terminate, else go to step 3b.

Let \mathbf{w} denote the multi-strategy that is achieved upon termination of the above algorithm. It can be readily seen that \mathbf{w} is feasible and all states up to some threshold \bar{i} are used with probability 1 (each state $i \leq \bar{i}$, $i \in \mathcal{H}$ is used by a single user with probability 1), while the remaining states are not used at all. The 0 – 1 property follows from step 3d. The threshold state phenomenon can be easily proved as otherwise one obtains a contradiction with the optimality of the \mathbf{w}^k in step 3c for some k .

4. Let $\bar{\mathbf{p}}$ be a NE such that \mathbf{w} has the same transmission probability assignment with $\bar{\mathbf{p}}$ for states $i \leq \bar{i}$. Such $\bar{\mathbf{p}}$ is guaranteed to exist by considering a reduced game played at states $i > \bar{i}$ with power budgets $\bar{P}^m - \sum_{i=\bar{i}}^h \pi_i w_i^m$, $m \in \mathcal{M}$.

We next show that the efficiency loss between $\hat{\mathbf{p}}$ and any $\bar{\mathbf{p}}$ is bounded by some fraction of $u(\hat{\mathbf{p}})$, where $\hat{\mathbf{p}}$ is the initial optimal threshold solution. To that end we consider the efficiency loss incurred in the transition from $\hat{\mathbf{p}}$ to $\bar{\mathbf{p}}$ through the path $\hat{\mathbf{p}} \rightarrow \mathbf{q} \rightarrow \mathbf{w} \rightarrow \bar{\mathbf{p}}$.

$\hat{\mathbf{p}} \rightarrow \mathbf{q}$: Note that $u(\mathbf{q}) \geq u(\hat{\mathbf{p}})$, since

$$u(\hat{\mathbf{p}}) = \sum_{i \in \mathcal{H}} \pi_i \sum_l \hat{p}_i^l (R_i^l \prod_{k \neq l} (1 - \hat{p}_i^k) - \lambda^l) \quad (62)$$

$$\leq \sum_{i \in \mathcal{H}} \pi_i \sum_l \hat{p}_i^l (R_i^l \prod_{k \neq l} (1 - \hat{p}_i^k) - \lambda^l) \quad (63)$$

$$\leq \sum_{i \in \mathcal{H}} \pi_i \sum_l \hat{p}_i^l (R_i^l - \lambda^l) \quad (64)$$

$$\leq \sum_{i \in \mathcal{H}} \pi_i (R_i^{m_i} - \lambda^{m_i}) \sum_l \hat{p}_i^l \quad (65)$$

$$\leq \sum_{i \in \mathcal{H}} \pi_i (R_i^{m_i} - \lambda^{m_i}) = u(\mathbf{q}) \quad (66)$$

where $\sum_{l \in \mathcal{M}} \hat{p}_i^l \leq 1$ and $\hat{p}_i^m \leq \hat{p}_i^{m_i}$ for all $m \in \mathcal{M}$, $i \in \mathcal{H}$. The existence of \hat{p}_i^m for all $m \in \mathcal{M}$, $i \in \mathcal{H}$ satisfying the first inequality follows by considering the aggregate utility maximization problem for each state $i \in \mathcal{H}$ separately and using the fact that at optimal utilization \mathbf{p} , $\sum_l p_i^l \leq 1$, as Proposition 1 suggests.

$\mathbf{q} \rightarrow \mathbf{w}$: For any user $m \in \mathcal{M}$ if $p_i^k = 0$ for $k \neq m$ whenever $p_i^m > 0$, then $u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \sum_{i \in \mathcal{H}} \pi_i p_i^m (R_i^m - \lambda^m)$, hence the payoff is weighted linear combination of power invested in different states. Now due to linearity it follows that assuming $\sum_{i \in \mathcal{H}} \pi_i p_i^m \geq \beta$ if user m utilizes a strategy $\tilde{\mathbf{p}}^m$ such that transmission probabilities in states $i > j$ for a fixed j are set to zero and $\sum_{i=j+1}^h \pi_i p_i^m \leq \alpha$, then $u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\tilde{\mathbf{p}}^m, \mathbf{p}^{-m}) \leq \frac{\alpha}{\beta} u^m(\mathbf{p}^m, \mathbf{p}^{-m})$ since α amount of power which is utilized in lower weights contribute at most $\frac{\alpha}{\beta} u^m(\mathbf{p}^m, \mathbf{p}^{-m})$ to user's payoff.

In step 3 of the algorithm observe that modifying actions of a user does not affect the payoff of other users. Let u^m be the initial payoff of user m in this step, then $\sum_{m \in \mathcal{M}} u^m = u(\mathbf{q})$. Denote payoff of user m after step 3c by \hat{u}^m , it follows that $\hat{u}^m \geq u^m (1 - \frac{\pi_{max}}{P_{min}})$ since for users satisfying $\Delta \bar{P}^m = 0$ the payoff actually increases when playing the best response. Also, for users with $\Delta \bar{P}^m > 0$ at least P_{min} amount of power is invested in the system and these users stop investing π_{max} amount of power in their worst states (as in (61)) in step 3c. Then playing best response the aggregate utility can only increase and it is larger than $u^m (1 - \frac{\pi_{max}}{P_{min}})$.

Similarly in step 3d of algorithm every user m using a state partially invests at least P_{min} amount of power and he stops investing at most π_{max} amount of power in his worst states. Denote his final payoff by \bar{u}^m . Then, $\bar{u}^m \geq \hat{u}^m(1 - \frac{\pi_{max}}{P_{min}}) \geq u^m(1 - \frac{\pi_{max}}{P_{min}})^2$. Since users who do not use any state partially do not modify their policies, it follows that

$$u(\mathbf{q}) \left(1 - \frac{\pi_{max}}{P_{min}}\right)^2 = \sum_{m \in \mathcal{M}} \left(1 - \frac{\pi_{max}}{P_{min}}\right)^2 u^m \leq \sum_{m \in \mathcal{M}} \bar{u}^m = u(\mathbf{w}) \quad (67)$$

$\mathbf{w} \rightarrow \bar{\mathbf{p}}$: Finally, it can be seen that

$$u(\mathbf{w}) \leq u(\bar{\mathbf{p}}). \quad (68)$$

since $\bar{p}_i^k = w_i^k$ for $i \leq \bar{i}$, $k \in \mathcal{M}$ and the contribution of remaining states to the aggregate utility can not be negative as in this case at least one user can improve his payoff by setting transmission probability in states $i > \bar{i}$ equal to zero and this contradicts with the fact that $\bar{\mathbf{p}}$ is a NE.

To summarize,

$$u(\hat{\mathbf{p}}) \left(1 - \frac{\pi_{max}}{P_{min}}\right)^2 \leq u(\mathbf{q}) \left(1 - \frac{\pi_{max}}{P_{min}}\right)^2 \leq u(\bar{\mathbf{p}}) \quad (69)$$

Hence,

$$\frac{u(\hat{\mathbf{p}})}{u(\bar{\mathbf{p}})} \leq \frac{1}{\left(1 - \frac{\pi_{max}}{P_{min}}\right)^2} \quad (70)$$

as the claim suggests. \square