Noncooperative Load Balancing in the Continuum Limit of a Dense Network

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***Abstract*—In transportation network research, the main approach for predicting traffic distribution due to noncooperative vehicle choices has been through fluid type models. The basic model considers a continuum of infinitesimal "non-atomic" vehicles, each seeking the shortest path to its destination. The resulting equilibrium turns out to be much simpler to characterize in comparison to the finite-vehicle case, yet provides a good approximation to the latter. A less familiar fluid-type model uses a continuum limit for the network topology. The limit network is a continuum plane which inherits its cost structure from the original network, and the corresponding equilibrium is identified as the *continuum traffic equilibrium*.**

This paper considers a similar equilibrium notion in a framework of a load balancing problem involving two processors, each requiring non-negligible workload (or "flow") to be handled by network resources. Besides a congestion cost at each resource (which is identical to both processors), each resource induces a processor-dependent connection cost, which is a function of its geographic location. The processors autonomously route their flow onto the different resources, with the objective of minimizing (non-cooperatively) their total cost. Assuming that the number of resources is relatively large, we apply the continuum approximation within a line (or bus) topology and study the Nash equilibria of the processor interaction. This approximation enables us to explicitly characterize the equilibrium in several cases and to obtain insights on its structure, including tight bounds on the efficiency loss due to noncooperation.

I. Introduction

This paper studies a distributed load balancing problem in the presence of a large number of resources. We consider two processors connected through a fast bus of some fixed length, which also connects the two processors to a large number of resources. Each processor divides a given workload (or "flow") between all available resources in order to minimize its operation cost, which consists of the delay incurred by sending flow over the network (proportional to the distance of the resource from the processor and/or to the number of hops the flow traverses), and the processing time at the resource itself. Motivating examples are multi-processor systems, in which each processor determines whether a task should be processed locally or scheduled to a different node for remote processing (see, e.g., [23], [16]). Other examples include novel communication link architectures that connect between processor nodes to input-output nodes, such as storage devices (e.g., InfiniBand [18]).

When each processor applies load balancing to minimize its own cost, a noncooperative game situation arises, analogous to interactions in network routing games with congestion costs (see [21] and references therein). While network routing games have been analyzed extensively, noncooperative load balancing has received less attention. The existing work in the latter area mostly focuses on models in which users are restricted to use only two resources, and where delay costs are location-independent (see Refs. [39]-[43] in [1]). A common challenge in the analysis of the problem is dealing with the complexity involved in characterizing the load balancing (Nash) equilibria.

In this paper, we consider load balancing in a dense network and show that when it is approximated by its continuum limit (i.e., there is a continuum of resources), a tight characterization of Nash equilibria becomes possible. In particular, we show how to calculate the equilibrium point, and derive closed-form solutions for special cost structures. We also establish bounds on the inefficiency of the equilibria (i.e., "price of anarchy" [21]) for this class of load-balancing games.

Related Literature. We survey below the relevant literature on the continuum traffic equilibrium, assumed to be unfamiliar to the bulk of the networking community. The continuum topology paradigm replaces a network of links and nodes by a continuum, where each point can be reached by every other point. This paradigm has been recently introduced into telecommunications networks with the pioneering paper of Jacquet [14]. This paper studies centralized routing in dense ad-hoc networks, using tools from geometrical optics. Based on the continuum approximation, methods that employ electrostatics tools were exploited in [25], [24] for studying other variants of the routing problem (see also [26] and references therein). The routing problem in the above references incorporates cost density functions that are related to energy consumption or to the density of mobiles that is required to satisfy a given demand. More recently, [13] has proposed a continuum topology approach in order to maximize the network lifetime.

The continuum topology methodology in dense ad-hoc networks has emerged independently of the existing theory of routing in massively dense networks, which was developed within road-traffic research. Already in 1952, Wardrop [27] and Beckmann [6] have introduced the continuum topology approximation in a competitive routing context, where not only the network becomes a continuum, but also the vehicles are represented as a continuum population of decision makers. This way of modeling competition in routing games has been an active research area among that community, see [9], [10], [12], [28] and references therein. Building upon these tools, as well as on optimal control methods, some characterization and numerical studies of the equilibria in ad-hoc networks have been recently proposed in [2], [3], [22].

Content and Contribution. In the present paper we introduce the continuum topology approach to study the equilibrium properties in load balancing scenarios. We consider a non-cooperative game restricted to two players (in contrast to the infinite player formalism of road traffic), where each player controls a splittable flow (that may represent a large number of packets or jobs). Our main contributions are summarized below.

- Focusing on identical resources and linear connection costs, we show how to calculate the Nash equilibrium by distinguishing between light, medium and heavy load conditions.
- We provide explicit equilibrium characterization results for polynomial congestion costs.
- We investigate the efficiency properties of Nash equilibria and provide a tight efficiency loss bound of 1.0818 due to selfish behavior for the case of linear congestion costs and symmetric users.
- In the appendix, we show that the continuum approximation is the limit of the discrete resource case, thereby justifying the use of the approximation (the precise statement of this result and the technical proof thereof are omitted here due to lack of space).

The structure of this paper is as follows. The network model is presented in Section II. Basic equilibrium properties are summarized in Section III. We then concentrate on the case of identical resources where connection costs are linear with distance. In Section IV we show how to calculate the equilibrium point for that case. In Section V, we focus on resources with polynomial congestion costs, and demonstrate the advantage of using the continuum approximation in comparison to the finite-resource formulation. We quantify in Section VI the efficiency loss due to selfish behavior for the case of linear congestion costs. Conclusions and future research directions are highlighted in Section VII.

II. The Model

We consider two processors (or users) that are connected through a link of length L. The same link also connects the two processors to additional resources. The flow demand of the processors is assumed fixed, and is given by r_i, $i = 1, 2$. Each processor divides its flow between the available resources in order to minimize its cost, which is defined below.

The cost for shipping a unit of flow through a given resource consists of two elements (i) a *connection* cost, which is proportional to the distance from the processor to the resource; (ii) a *congestion* cost, which depends on the total flow handled by the resource.

Each resource is modeled as a link, and is identified, without loss of generality, by its distance from processor-1 (see Figure 1). That is, link x is located x length-units from processor-1 and $L-x$ length units from processor-2. Let $u^i(x)$ denote the flow that processor-i ($i = 1, 2$) sends to link x; further denote

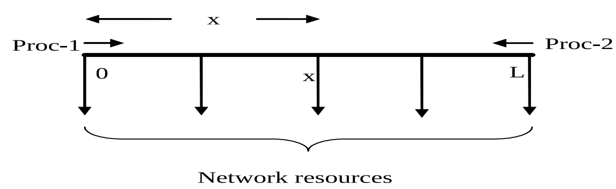

Fig. 1. The two processor network. Under the continuum approximation there is an outgoing link for every $x \in [0, L]$.

by $u(x) = u^1(x) + u^2(x)$ the total flow at that link. The per-unit connection cost to link x is given by the function $f^i(x)$ (note that this cost is user-dependent). The congestion cost per unit flow at link x is given by the function $g_x(u^1(x)+u^2(x)) \equiv g_x(u(x))$.

In this paper we study the *continuum* noncooperative game that corresponds to the above model. In the continuum case it is assumed that there exists a resource at every $x \in [0, L]$. A formal definition of the game is provided below.

- The set of users (processors) is $\{1, 2\}$.
- User strategies are given by functions $u^i : [0, L] \to [0, \infty)$. A feasible strategy for user i is a flow allocation that obeys

$$\int_0^L u^i(x)dx = r_i. \quad (1)$$

- User costs are given by

$$J^i(u^1, u^2) = \int_0^L u^i(x) \left(f^i(x) + g_x(u(x))\right) dx. \quad (2)$$

For now, we assume that the integrals in (1) and (2) exist, and justify this assumption later in Section IV. We refer to the game between the two processors as the *load balancing game*.

Let $\mathbf{u}^i$ denote the strategy of user i. A Nash Equilibrium (NE) [11] of the load balancing game is a joint flow allocation $(\mathbf{u}^1, \mathbf{u}^2)$ which is feasible for both processors and satisfies the following condition:

$$J^1(\mathbf{u}^1, \mathbf{u}^2) \le J^1(\tilde{\mathbf{u}}^1, \mathbf{u}^2) \text{ and } J^2(\mathbf{u}^1, \mathbf{u}^2) \le J^2(\mathbf{u}^1, \tilde{\mathbf{u}}^2) \quad (3)$$

for any feasible flow allocations $\tilde{\mathbf{u}}^i$, $i = 1, 2$. In particular, the NE is a network operating point which is stable in the sense that neither processor finds it beneficial to unilaterally change its flow allocation.

Throughout the paper we assume that the functions $f^i(x)$ and $g_x(u(x))$ satisfy the following conditions:

Assumption 1: The function $f^i(x)$ is continuous in x and strictly increasing with the distance from the processor to the resource for $i = 1, 2$, i.e., $f^1(x)$ is strictly increasing in x while $f^2(x)$ is strictly decreasing in x.

Assumption 2: The function $g_x(u)$ is positive, continuous, strictly increasing and convex in u, with $g_x(0) = 0$.

III. Basic Equilibrium Properties

We start our analysis of the load balancing game by providing a basic characterization result for the Nash equilibrium point, using the first-order optimality conditions for the individual optimization problem of the players (given the equilibrium strategy of the other player). In view of the separable structure of this optimization problem, the optimality conditions decouple over each $x \in [0, L]$[1].

Lemma 1: Let $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$ be a NE of the load balancing game. Then, for all $i = 1, 2$, there exists a scalar $\lambda^i > 0$ such that the following conditions hold for all $x \in [0, L]$ with probability one:

$$\begin{cases} f^i(x) + g_x(u(x)) + u^i(x) g'_x(u(x)) = \lambda^i & u^i(x) > 0, \\ f^i(x) + g_x(u(x)) + u^i(x) g'_x(u(x)) \geq \lambda^i & u^i(x) = 0. \end{cases} \quad (4)$$

Proof: Given the routing allocation of the other processor, each processor wishes to minimize its overall cost (2) subject to the following constraints: (i) a total flow constraint (1); (ii) a positivity constraint at each link x, namely $u^i(x) \geq 0$. The Lagrangian of user i is thus the following:

$$L^i(\mathbf{u}, \lambda^i, \mu^i) = \int_0^L u^i(x) \left(f^i(x) + g_x(u(x)) \right) dx + \lambda^i (r_i - \int_0^L u^i(x) I_x) dx - \int_0^L \mu^i(x) u^i(x) I_x dx,$$

where $\mu^i(x)$ is the Lagrange multiplier that corresponds to the positivity constraint at point x. The KKT optimality conditions (which are sufficient and necessary in our case, due to the convexity of g_x) suggest that the derivative of the above Lagrangian with respect to $u^i(x)$ (at any point x) should be equal to zero. This leads to the following equation.

$$\frac{\partial L^i(\mathbf{u}, \lambda^i, \mu^i)}{\partial u^i(x)} = f^i(x) + g_x(u(x)) + u^i(x) g'_x(u(x) - \mu^i(x) = \lambda^i.$$

Conditions (4) are obtained by incorporating the Complementary Slackness property (i.e., $\mu^i(x) u^i(x) = 0$) and noting that $\mu^i(x) \geq 0$. The fact that $\lambda^i > 0$ follows from the observation that the KKT conditions (4) are met with equality for at least some $x \in [0, L]$, whereas the left hand side of the associated KKT equation is strictly positive by Assumption 2.

Due to decoupling over x, the above KKT optimality conditions hold point-wise. In the continuum case, we notice, however, that each processor could change his strategy on a zero measure set without affecting its cost or the other processor's cost. Hence, the above lemma applies under the restriction that the conditions hold with probability one (i.e., everywhere in $[0, L]$ except a measure zero set). □

The optimality conditions (4) would serve in subsequent sections as a main building block for characterizing the Nash equilibria of the load balancing game. An immediate result due to these conditions is the uniqueness of the equilibrium point, which essentially follows from the uniqueness of the Nash equilibrium for (finite) parallel-link networks, obtained in [20].

Theorem 1: In case that a Nash equilibrium (NE) exists, the load balancing game has a unique NE with probability one, i.e., if $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are two NE of the load balancing game [cf. Eq. (3)], then $u^i(x) = \tilde{u}^i(x)$ for all $i = 1, 2$ and all $x \in [0, L]$ except on a measure zero set.

Proof: (outline) Using the notations of [20], the overall cost at each link is of Type 'A'. Thus uniqueness for the finite-link case holds by Theorem 2.1 in [20]. The proof of that theorem relies on the KKT conditions; using these conditions, it is shown that the set of links, for which the total flow differs when comparing two equilibria, is an empty set. Since the same KKT conditions hold (with probability 1) by Lemma 1, we may follow the lines of Theorem 2.1 in [20] (replacing summation over links by integrals) to establish that the uniqueness result holds for the continuum approximation (up to sets of links with measure zero). A complete proof can be found in the appendix. □

From a game-theoretic perspective, the infinite-dimensional action space of each user requires special care with regard to the *existence* of an equilibrium point, which does not follow straightforwardly as in finite resource case (see, e.g., [20]). In the full version of this work [4], we establish sufficient conditions for the existence of an equilibrium. In the current paper, we advocate a direct approach, by explicitly characterizing the equilibrium point for a given cost-structure (see Section V), hence establishing its existence.

IV. Continuum of Identical Resources with Linear Connection Costs

We henceforth consider the case where the connection cost is linearly proportional to the distance the flow traverses from the processor to the resource. More precisely, $f^1(x) = \alpha x$ and $f^2(x) = \alpha(L - x)$, where $\alpha > 0$ is the connection cost per length-unit for shipping a unit of flow. The linearly-dependent delay can be interpreted as representing propagation delay (which is proportional to distance), or delay due to relaying (which linearly depends on the number of relays, or the length of the interval under our continuum approximation). Alternately, α may stand for the price for using the fast link in price-based networks. Resources are identical in terms of their available resources and associated costs. Hence, a unified function $g = g_x$, $x \in [0, L]$ (which obeys Assumption 2) will be used for the congestion cost. In addition, we assume that many resources are available, hence approximate this scenario by assuming a continuum of resources.

Throughout the paper, we denote by x_1 the farthest location (relative to processor-1 origin) at which processor-1 stops transmitting alone. Similarly, x_2 denotes the farthest location (relative to processor-2 origin) at which processor-2 stops transmitting alone.

[1] In the sequel, the notation $g'_x(u(x))$ [or $g'(u(x))$] denotes a subgradient of the convex function g_x [or g] at the scalar $u(x)$ for a given strategy $u(x)$ and some $x \in [0, L]$, see [7].

A. Basic Properties

We establish below an important equilibrium property for linear connection costs: the total flow at every link in which both processors transmit is the same; it does not depend on x.

Lemma 2: Denote by $\mathcal{X} \subset [0, L]$ the subset of links in which both users transmit at equilibrium. Then $u(x_1) = u(x_2) \stackrel{\triangle}{=} \bar{u}$ for every $x_1, x_2 \in \mathcal{X}$

Proof: Summing the KKT optimality conditions (4) of both processors, we obtain a single equation for every link x,

$$\alpha L + 2g(u(x)) + u(x)g'(u(x)) = \lambda, \tag{5}$$

where $\lambda = \lambda^1 + \lambda^2$. Define $h(u(x)) = 2g(u(x)) + u(x)g'(u(x))$. Then (5) is equivalent to $h(u(x)) = \lambda - \alpha L$. The right hand side of this equation does not depend on x. Hence, the left hand side does not depend on x as well. Furthermore, $h(u(x))$ is strictly increasing in $u(x)$ (by Assumption 2). This suggest that there exists a unique (and identical) value for every $u(x)$ (denoted $\bar{u}$) such that (5) is met. □

It is easy to verify that the region in which both processors transmit is a compact subset in $[0, L]$. Based on the above lemma, we may derive the flow of each processor at the commonly shred links as a function of the flow of the closed point to the processor for which both processors transmit. This is summarized in the next lemma.

Lemma 3: Let x_1 and x_2 be the closest points to processors 1 and 2 respectively at which both processors transmit. Then,

$$u^1(x) = u^1(x_1) - \frac{\alpha(x - x_1)}{g'(\bar{u})}, \quad x \in [x_1, x_2], \tag{6}$$

$$u^2(x) = u^2(x_2) - \frac{\alpha(x_2 - x)}{g'(\bar{u})}, \quad x \in [x_1, x_2]. \tag{7}$$

Proof: We show the claim for processor-1, where the equation for processor-2 follows similarly. Consider the KKT equations of processor-1 at location x_1 and at location x, namely $\alpha x_1 + g(\bar{u}) + u^1(x_1)g'(\bar{u}) = \lambda^1$, and $\alpha x + g(\bar{u}) + u^1(x)g'(\bar{u}) = \lambda^1$. Subtracting these two equations leads to (6). □

We note that the above two lemmas do not require the continuum approximation, i.e., the same results hold in the finite-link case. However, it is hard to exploit them for the latter case, as we demonstrate in Section V-C. Two important observation that are based on Lemma 3 are in place:

1) The flow of each processor in the commonly shared region varies *linearly*, regardless of the structure of the congestion cost g.
2) In order to calculate the flow of each processor in every x of the commonly shared region, we require the locations x_1, x_2, the flows at these two locations and the total flow in each point $\bar{u}$.

For the analysis of the equilibrium point, we distinguish between three different scenarios of traffic load: (i) Light load (Section IV-B), where every processor uses the links that are nearer to it, and consequently the traffic of the two processors does not overlap; (ii) Medium load (Section IV-C), where the processors overlap on a subset of links; (iii) Heavy load (Section IV-D), where both processors use all links. In Section V we consider a specific congestion-cost function, for which we obtain the conditions for the network being at any of the above cases, leading to an explicit characterization of the equilibrium point.

B. Light Load Conditions

1) Separability:

Definition 4.1: A flow allocation $\mathbf{u}$ is *separable* if each processor sends flow to links that are located near it, so that the flows of the two processors do not interact. More precisely, there are constants $0 < x_1 < x_2 < L$ such that $u^1(x) = 0$ for $x > x_1$ and $u^2(x) = 0$ for $x < x_2$. A *separable equilibrium* is an equilibrium where the flow allocation $\mathbf{u}$ is separable.

Let $\widehat{\mathbf{u}}$ be the optimal flow allocation for each processor pretending that the other processor does not exist. $\widehat{\mathbf{u}}^i$ is thus processor-i's flow that minimizes $J^i(\mathbf{u})$ for $\mathbf{u}^j = \mathbf{0}$, $j \neq i$.

Proposition 2: (i) The flow allocation $\widehat{\mathbf{u}}$ is separable if and only if it is a separable equilibrium.
(ii) Consequently, the flow allocation $\mathbf{u}$ is a separable equilibrium if and only if it is a separable optimal (centrally-assigned) flow allocation.

Proof: (i) (a) ($\widehat{\mathbf{u}}$ is separable $\Rightarrow$ separable equilibrium). For any $\mathbf{u}^i$ we have $J^i(\widehat{\mathbf{u}}) = J^i(\widehat{\mathbf{u}}^i, \mathbf{0}) \leq J^i(\mathbf{u}^i, \mathbf{0}) \leq J^i(\mathbf{u}^1, \widehat{\mathbf{u}}^2)$ (The first equality follows from the separability assumption, the first inequality follows from the optimality of $\widehat{\mathbf{u}}$, and the second inequality follows from the monotonicity of g). This implies that (a) holds by the definition of the NE (3).

(b) ($\mathbf{u}$ is a separable equilibrium $\Rightarrow$ $\mathbf{u} = \widehat{\mathbf{u}}$). If $\mathbf{u}^1$ is a separable equilibrium, then it is the optimal flow distribution for processor-1 (for sending a total flow of r_1) in a subnetwork which contains the links in the region $[0, x_2]$ only, for which processor-2 does not ship flow. Indeed, if it were not, then processor-1 could modify its flow distribution within that region to improve its cost, in contradiction to being at a NE. The addition of links at locations $[x_2, L]$ will obviously not cause a change in the optimal flow allocation, since those are links with higher cost, being far from processor-1 compared to the links in $[x_1, x_2]$ that are in use.

(ii) We prove that $\widehat{\mathbf{u}}$ is separable if and only if it is a separable optimal allocation. The claim would then follow by applying (i). Let $\mathbf{u}_*$ be the optimal flow allocation (unique up to a measure-zero set by the strict convexity of the associated optimization problem).

(a) $\widehat{\mathbf{u}}$ is separable $\Rightarrow$ separable optimum). The proof proceeds similar to (i)(a). For any $\mathbf{u}^i$ we have

$$J^i(\widehat{\mathbf{u}}) = J^i(\widehat{\mathbf{u}}^i, \mathbf{0}) \leq J^i(\mathbf{u}^i, \mathbf{0}) \leq J^i(\mathbf{u}^1, \mathbf{u}^2).$$

This implies that $\sum_i J^i(\widehat{\mathbf{u}}) \leq J^i(\mathbf{u}^1, \mathbf{u}^2)$ for every $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$. Consequently, $\sum_i J^i(\widehat{\mathbf{u}}) = J^i(\mathbf{u}^1_*, \mathbf{u}^2_*)$. Hence, $\mathbf{u}_* = \widehat{\mathbf{u}}$ with probability 1.

(b) ($\mathbf{u}_*$ is a separable optimal allocation $\Rightarrow$ $\mathbf{u} = \widehat{\mathbf{u}}$). The proof follows by an almost-identical argument as in (i)(b); we thus state it briefly: Note that $\mathbf{u}^1$ is an optimal flow allocation for user 1 over the set of links $[0, x_2]$; hence, trivially, it is also is an optimal flow allocation for user 1 over the set of links $[0, x_2]$ pretending user 2 does not exist. The addition of links $[x_2, L]$ would not modify this allocation, since these are farther than the links $[x_1, x_2]$ which are decided not to be used. $\square$

We note that the above proposition is valid for general continuous increasing connection costs, as long as congestion cost functions are identical (i.e., $g_x = g$).

2) Characterization of the equilibrium point: The KKT optimality conditions for this setup are as in (4), except that $u(x)$ is replaced everywhere by $u^i(x)$. Consequently, we obtain the following equation:

$$g(u^i(x)) + u^i(x)g'(u^i(x)) = \lambda^i - \alpha\Delta^i(x), \tag{8}$$

where $\Delta^1(x) = x$ and $\Delta^2(x) = L - x$. Define

$$h(u(x)) \stackrel{\triangle}{=} g(u^i(x)) + u^i(x)g'(u^i(x)). \tag{9}$$

The function $h(u(x))$ is a continuous increasing function by Assumption 2. Consequently, its inverse h^{-1} is a well-defined increasing function. Hence,

$$u(x) = h^{-1}\big(\lambda^i - \alpha\Delta^i(x)\big). \tag{10}$$

Note that x_i is the location at which processor-i stops transmitting. We next show that the flow at the locations x_1 and x_2 equals to zero.

Lemma 4: Consider light load conditions. Then $u^i(x_i) = 0$, $i = 1, 2$.

Proof: For simplicity, we shall consider only processor-1. Assume by contradiction that $u^1(x_1) > 0$. Then for $\epsilon > 0$ small, $\alpha x_1 + g(u^1(x_1)) + u^1(x_1)g'(u^1(x_1)) > \alpha(x_1 + \epsilon) + g(u^1(x_1 + \epsilon)) + u^1(x_1 + \epsilon)g'(u^1(x_1 + \epsilon)) = \alpha(x_1 + \epsilon)$. The last inequality obviously contradicts the KKT optimality conditions (4). $\square$

Using Lemma 4, it follows that $\lambda^1 = \alpha x_1$ and $\lambda^2 = \alpha(L - x_2)$. Thus,

$$u(x) = h^{-1}\big(\alpha\Delta^i(x_i) - \alpha\Delta^i(x)\big). \tag{11}$$

The quantities x_1, x_2 can be calculated by using the feasibility constraint (1). Using (10), the following equation are valid for processor-1 and processor-2

$$\int_0^{x_1} h^{-1}(\alpha x_1 - \alpha x)dx = r_1, \tag{12}$$

$$\int_{x_2}^{L} h^{-1}(\alpha x - \alpha x_2)I_x dx = r_2. \tag{13}$$

Based on the above equations, explicit flow formulae can be obtained for specific cost functions, such as $g(u) = (c-u)^{-1}$ (M/M/1) and $g(u) = cu^\beta$ (polynomial), on which we focus further in Section V.

C. Medium Load Conditions

In this subsection we consider cases where there is a partial overlap between the flows of the two processors. Two different scenarios are possible: (i) Each of the two processors uses a subset of links; (ii) One of the two processors uses a subset of links, whereas the other processor uses all links.

1) Both processors use a subset of links: The unknown variables in this case, which are needed in order to characterize the equilibrium point, are x_1, x_2 and $\bar{u}$. This property follows by noting that $u^1(x_1) = u^1(x_2) = \bar{u}$. The above variables may be obtained through a set of three equations, as we specify below.

The first equation compares the marginal cost of one of the processors (say processor-1) at points x_1 and x_2. The marginal costs at these points are equal since both processors transmit at these points. We assert that the flow of processor-1 at x_2 is zero, and so is the flow of processor-2 at x_1. This claim follows by the continuity of the flow allocation at the regions in which each processor transmits alone (see Eq. (11)) and then by using similar arguments as in Lemma 4. Consequently, we obtain the following equation: $\alpha x_1 + g(\bar{u}) + \bar{u}g'(\bar{u}) = \alpha x_2 + g(\bar{u})$, i.e.,

$$\alpha x_1 + \bar{u}g'(\bar{u}) = \alpha x_2. \tag{14}$$

The next equations consider the total flow of each processor. Note that $\lambda^1 = g(\bar{u}) + \alpha x_2$, $\lambda^2 = g(\bar{u}) + \alpha(L - x_1)$. In addition, the flow distribution of each processor includes a subregion in which it submits alone and a commonly shared subregion. Using Lemma 2 and the analysis for the light network conditions (Section IV-B), we obtain the following equations:

$$\int_0^{x_1} h^{-1}(\alpha x_2 - \alpha x + g(\bar{u}))dx + \frac{\bar{u}}{2}(x_2 - x_1) = r_1, \tag{15}$$

$$\int_{x_2}^{L} h^{-1}(\alpha x - \alpha x_1 + g(\bar{u}))dx + \frac{\bar{u}}{2}(x_2 - x_1) = r_2.$$

We observe that both processors submit the same amount of flow in the commonly shared region. This can be seen by noting that the flow of both processors is $\bar{u}$ at the closest point of the commonly shared region, and linearly decreases to zero by Lemma 3 (see Figure 3 for illustration).

2) One processor uses all links: Assume that processor-1 is the processor that uses all links, while processor-2 is the one that uses only a subset thereof. If it is the other way around, similar equations as the one we present below can be obtained by symmetry. For this setup,

$$\lambda^1 = \alpha x_1 + g(\bar{u}) + \bar{u}g'(\bar{u}). \tag{16}$$

Note that we remain with two variables: $\bar{u}$ and x_1. If these are obtained, the flow of each user in every link can be calculated by using (10) and Lemma 3.

Consider the point $x = L$. Using Lemma 3, the flow of processor-1 at that point is given by $u^1(L) = \bar{u} - \frac{\alpha(L-x_1)}{g'(\bar{u})}$; hence,

$$u^2(L) = \frac{\alpha(L - x_1)}{g'(\bar{u})}. \tag{17}$$

Using (10), (16) and (17), we obtain the following equations:

$$\int_0^{x_1} h^{-1}(\alpha x_1 - \alpha x + g(\bar{u} + \bar{u}g'(\bar{u})))dx + (L - x_1)\bar{u} - r_2 = r_1,$$

$$\frac{\alpha(L - x_1)}{2g'(\bar{u})}(L - x_1) = r_2,$$

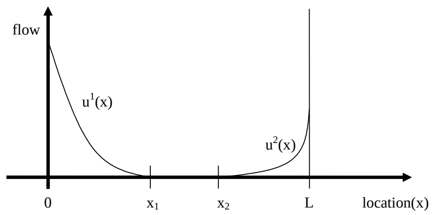

Fig. 2. Light load conditions. Each processor uses a different subset of links.

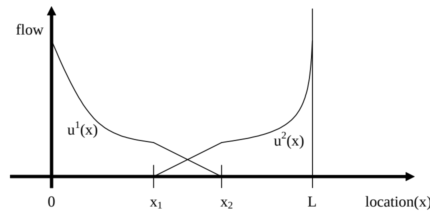

Fig. 3. Medium load conditions. Case 1: Both processors use a subset of links that is closer to their origin.

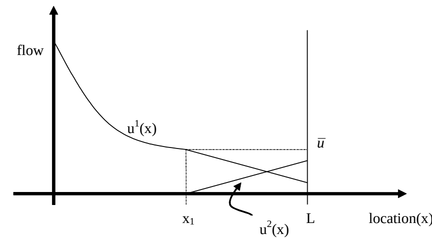

Fig. 4. Medium load conditions. Case 2: One of the processors (processor-1 in this figure) uses all links, while processor-2 uses only a subset of links that are closer to its origin.

where the second equation follows by noting that the flow integral of processor-2 is in the form of a triangle (see Figure 4).

D. Heavy Load Conditions

In the heavily loaded network setup, both processors use all links. Under the heavily loaded network assumption, Lemma 2 leads to quantifying the total flow at each link:

Proposition 3: Assume the network is heavily loaded. Then the total flow at every link is given by

$$\bar{u} = \frac{r_1 + r_2}{L}. \tag{18}$$

Proof: Immediate from Lemma 2 and (1). □

Based on Proposition 3, we may explicitly calculate the flow of the two processors at each link. For simplicity, we derive the flow equation for processor-1. The results for processor-2 follow by symmetry.

We use Lemma 3 with $x_1 = 0$. Substituting (6) in (1) yields the following equation $Lu^1(0) - \frac{\alpha}{2g'(\bar{u})}L^2 = r_1$. Hence, $u^1(0) = \frac{r_1+\beta(\bar{u})}{L}$, where $\beta(\bar{u}) = \frac{\alpha L^2}{2g'(\bar{u})}$. By substituting the last relation back in (6) we obtain the flow of processor-1 at every link x,

$$u^1(x) = \frac{r_1 + \beta(\bar{u})}{L} - \frac{\alpha x}{g'(\bar{u})}. \tag{19}$$

By symmetry, the flow of processor-2 is given by

$$u^2(x) = \frac{r_2 - \beta(\bar{u})}{L} + \frac{\alpha x}{g'(\bar{u})}. \tag{20}$$

Simple algebra leads to the following result.

Proposition 4: Consider heavy load conditions. The flows of the processors at every link x are given by

$$u^1(x) = \frac{r_1}{L} + \frac{\alpha(L-2x)}{2g'(\bar{u})}. \tag{21}$$

$$u^2(x) = \frac{r_2}{L} + \frac{\alpha(2x-L)}{2g'(\bar{u})}. \tag{22}$$

Notice that the ratio between the flows at $L/2$ is the same as the ratio between the processor demands.

Remark 1: It is of interest to examine here the effect of α on the total equilibrium cost. Increasing α is analogous to decreasing the network capacity, as connections become slower. While it is obvious that the cost increases under a centrally-controlled allocation, this need not be the case in a noncooperative regime (as shown by the well-known Braess' Paradox [8]). Proposition 3 leads to the observation that the total cost $J^1 + J^2$ does increase with α, as $\bar{u}$, and consequently the total congestion cost does not depend on α, while the connection cost increases with α. We emphasize that this property is valid under the heavily loaded network setup.

Remark 2: The equilibrium that we characterize in each of the three different load scenarios is seen to be continuous in each user's policy $\mathbf{u}^i$, hence justifying the integrability assumption made in (1)–(2).

V. Polynomial Congestion Costs

Let $g(u) = cu^\beta$, where $c > 0$. In this case, the function h (defined in (9)) is equivalent to $h(u) = cu^\beta(1+\beta)$; its inverse is given by

$$h^{-1}(u) = \gamma_c\left(\frac{u}{1+\beta}\right)^{\frac{1}{\beta}}, \tag{23}$$

where $\gamma_c \triangleq c^{-\frac{1}{\beta}}$. In Section V-A we focus on extreme load conditions and obtain explicit flow equations. In Section V-B we assume that the processor demands are equal. Using symmetry, we are able to obtain tractable expressions for medium load conditions as well. Moreover, we are able to fully characterize the equilibrium for linear latency functions, including mapping the given demands to the respective load conditions (light/medium/heavy).

A. Extreme load conditions (light/heavy)

Light Load. Condition (12) becomes

$$\gamma_c \int_0^{x_1} \left(\frac{\alpha x_1 - \alpha x}{1+\beta}\right)^{\frac{1}{\beta}} dx = \gamma_c \frac{\left(\frac{\alpha x_1 - \alpha x}{1+\beta}\right)^{\frac{1}{\beta}+1}}{\frac{1}{\beta}+1} \frac{1+\beta}{\alpha}\Big|_{x_1}^{0}$$

$$= \gamma_c \frac{\beta}{\alpha}\left(\frac{\alpha x_1}{1+\beta}\right)^{\frac{\beta+1}{\beta}} = r_1.$$

Hence, $x_1 = \frac{1+\beta}{\alpha}\left(\frac{\alpha}{\beta\gamma_c}r_1\right)^{\frac{\beta}{\beta+1}}$; similarly,

$$x_2 = \frac{1+\beta}{\alpha}\left(\frac{\alpha}{\beta\gamma_c}r_2\right)^{\frac{\beta}{\beta+1}}.$$

We see that for demands r_1, r_2 that obey the following condition

$$\frac{1+\beta}{\alpha}\left(\frac{\alpha}{\beta\gamma_c}\right)^{\frac{\beta}{\beta+1}}\left[r_1^{\frac{\beta}{\beta+1}}+r_2^{\frac{\beta}{\beta+1}}\right]\leq L, \tag{24}$$

the solution is separable and is thus an equilibrium according to Proposition 2. Having found x_1 and x_2, the flow of every user at each point can be obtained by (11) and (23).

Heavy Load. The flows at every point x are immediately obtained through (18) and replacing $g'(u) = c\beta u^{\beta-1}$ in (21)–(22).

B. The symmetric case and explicit equilibrium calculation

In what follows, we concentrate on medium load conditions, and assume $r_1 = r_2 = r$. Later, we use our results from Section V-A to fully calculate the equilibrium point for linear congestion costs.

By symmetry, we are left with two variables x_0 and $\bar{u}$, where $x_1 = x_0$ and $x_2 = L - x_0$. Equations (14) and (15) become $g'(\bar{u})\bar{u} = \alpha(L - 2x_0)$, and

$$2\int_0^{x_0} h^{-1}(\alpha(L-x_0)+g(\bar{u})-\alpha x)dx+(L-2x_0)\bar{u}=2r.$$

For $g(u) = cu^\beta$, the first equation is equivalent to

$$g(\bar{u})=c\bar{u}^\beta=\frac{\alpha}{\beta}(L-2x_0). \tag{25}$$

After some algebra, the second equation is given by

$$\frac{2\beta\gamma_c}{\alpha}\left[\left(\frac{\alpha(L-x_0)+g(\bar{u})}{1+\beta}\right)^{\frac{\beta+1}{\beta}}-\left(\frac{\alpha(L-2x_0)+g(\bar{u})}{1+\beta}\right)^{\frac{\beta+1}{\beta}}\right]+(L-2x_0)\bar{u}=2r. \tag{26}$$

Substituting (25) in (26) leads to a single equation in a single variable x_0,

$$2\frac{\beta\gamma_c}{\alpha}\left[\left(\frac{\alpha(L-x_0)+\frac{\alpha}{\beta}(L-2x_0)}{1+\beta}\right)^{\frac{\beta+1}{\beta}}-\left(\frac{\alpha(L-2x_0)+\frac{\alpha}{\beta}(L-2x_0)}{1+\beta}\right)^{\frac{\beta+1}{\beta}}\right]+(L-2x_0)\left(\frac{\alpha}{\beta c}(L-2x_0)\right)^{\frac{1}{\beta}}=2r. \tag{27}$$

It can be easily shown that the left hand side of (27) strictly increases with x_0. Hence, there exists a unique solution to that equation. However, for general β the solution can be obtained only numerically. We thus turn to explicitly solve (27) for $\beta = 1$, which corresponds to linear congestion costs. For $\beta = 1$, (27) becomes

$$\frac{2}{\alpha c}\left[\left(\frac{\alpha(L-x_0)+\alpha(L-2x_0)}{2}\right)^2-(\alpha(L-2x_0))^2\right]+\frac{\alpha}{c}(L-2x_0)^2=2r, \tag{28}$$

i.e., quadratic in x_0. After some algebra, we obtain the equation $x_0^2 - 4Lx_0 + 2L^2 - \xi = 0$, where $\xi \triangleq \frac{4rc}{\alpha}$. Solving this equation yields

$$x_0 = 2L - \sqrt{2L^2+\xi}. \tag{29}$$

We then obtain the second variable $\bar{u}$ via (25). These two variables are sufficient for finding the processor flows at any x.

To conclude the characterization of the Nash equilibrium, we next obtain explicit conditions for belonging to each of the scenarios (low/medium/heavy loads). We restrict ourselves to the symmetric case with linear congestion costs.

- The condition for light load (24) becomes $r \leq \frac{1}{16c}\alpha L^2$.
- Substituting $x_0 = 0$ in (29) gives us the highest value of r for which the network is under medium load conditions. We obtain the following condition $\frac{1}{16c}\alpha L^2 \leq r \leq \frac{1}{2c}\alpha L^2$.
- If $r > \frac{1}{2c}\alpha L^2$, we are thus under heavy load conditions.

C. Characterization without the Continuum Approximation

In this subsection we demonstrate the difficulties in calculating the equilibrium assignment without using the continuum approximation. We consider the simplest case of a single user under light-load conditions. Let $x_1 < x_2 < \cdots < x_N$ be the locations of the links, and assume $g(u(x)) = u^\beta$. Note the KKT conditions (4) may be applied for the finite case, and so is the definition of h^{-1} in (10). The total flow constraint (analogous to (12)) is given by

$$\sum_{j=1}^{K(r_1)} h^{-1}(\lambda^1-\alpha x_j)=\sum_{j=1}^{K(r_1)}\left(\frac{\lambda^1-\alpha x_j}{1+\beta}\right)^{\frac{1}{\beta}}=r_1,$$

where $K(r_1)$ is the last link used by user 1 (depending on its total demand r_1). Unlike in the continuum case, the Lagrange multiplier λ^1 cannot be expressed as a function of location (as Lemma 4 does not necessarily hold). Hence, λ^1 has to be extracted in order to calculate the link flows. Note that for the given example, λ^1 can be obtained only numerically. An additional major difficulty in characterizing the NE arises since the farthest link used by processor-1 (i.e., $K(r_1)$) is not known in advance. Hence, a search procedure is required in order to obtain $K(r_1)$, a quantity that is irrelevant under the continuum approximation.

VI. Efficiency Loss

In this section we briefly examine through a specific cost structure the extent to which selfish behavior affects system performance. That it, we are interested to compare the quality of the obtained equilibrium point to the centralized, system-optimal solution. Recently, there has been much work in quantifying the "efficiency loss" incurred by selfish behavior of users in networked systems (see [21] for a comprehensive review). Our ability to characterize the equilibrium point under the continuum approximation enables us to compare the performance at equilibrium to the optimal performance.

We focus in this section on linear connection and congestion costs, i.e., $f^1(x) = \alpha x$, $f^2(x) = \alpha(L - x)$ and

$g(u(x)) = cu(x)$. It has been established in [15] that the (unique) equilibrium point coincides with the social optimum for the case where there are no connection costs. We next show that the incorporation of connections costs may result in some efficiency loss, despite the linearity of both connection and congestion costs. We are able to upper-bound this efficiency loss by deriving the so-called price-of-anarchy (PoA) [21] of our network.

Naturally, we draw attention in our analysis to heavy load conditions, since it is expected that the overall equilibrium cost (i.e., the sum of costs of both processors) would get worse, compared to the optimal flow allocation, as congestion increases. Indeed, under light load conditions, it is obvious that the equilibrium point coincides with the globally optimal solution. We assume that both processors have an identical demand r. We calculate below both the equilibrium and optimal costs as a function of the parameters (α, c, r), and then derive the efficiency loss ratio.

Equilibrium Cost. Recalling that the total flow in each link is identical and given by $\frac{2r}{L}$, the total cost of user 1 is given by $J^1(\mathbf{u}) = \int_0^L u^1(x)\big(\alpha x + \frac{2r}{L}\big)$, where $u^1(x) = \frac{r}{L} + \frac{\alpha(L-2x)}{2c}$ due to (21). Thus,

$$\begin{aligned} J^1(u) &= \int_0^L \left(\frac{r}{L} + \frac{\alpha L}{2c} - \frac{\alpha x}{c}\right)\left(\frac{2rc}{L} + \alpha x\right) dx \\ &= \int_0^L \left(\frac{r}{L} + \frac{\alpha L}{2c}\right)\frac{2rc}{L} + \left(\frac{\alpha L}{2c} - \frac{r}{L}\right)\alpha x - \frac{\alpha^2 x^2}{c} dx \\ &= \frac{2r^2 c}{L} + \alpha r L + \frac{\alpha^2 L^3}{4c} - \frac{\alpha r L}{2} - \frac{\alpha^2 L^3}{3c} \\ &= \frac{2r^2 c}{L} + \frac{\alpha r L}{2} - \frac{\alpha^2 L^3}{12c}. \end{aligned}$$

Consequently, the total cost at equilibrium is given by

$$JE = 2J^1(\mathbf{u}) = \frac{4r^2 c}{L} + \alpha r L - \frac{\alpha^2 L^3}{6c}. \tag{30}$$

Optimal Cost. The basic property that allows us to derive the optimal cost is that under heavy load conditions, an optimal cost is obtained when each user sends flow to links up to distance of $L/2$.

Proposition 5: Consider users with identical flow demands r under heavy-load conditions and a continuum of identical resources. Then the optimal (centrally-assigned) flow allocation is obtained when both processors use the links up to a distance of $L/2$ from their origin.

Proof: Note first that if the optimal allocation is separable (see Definition 4.1) then the Nash equilibrium is separable by Proposition 2(ii), in contradiction to the heavy load conditions. Hence, all network links are used in an optimal allocation. By symmetry considerations, an optimal allocation $\mathbf{u}$ obeys $u(x) = u(L-x)$ for every $x \in [0, L]$ (with probability 1). Hence

$$g(u(x)) = g(u(L-x)). \tag{31}$$

Assume by contradiction that an optimal allocation assigns a positive flow from processor-1 demand to a subset $S_1 \subset [L/2, L]$ of measure greater than zero. Then by symmetry, the optimal allocation assigns a positive flow from processor-2 demand to a subset $S_2 \subset [0, L/2]$ of measure greater than zero, where $x \in S_1 \rightarrow (L-x) \in S_2$ with probability 1. Due to (31), we can improve the above assignment by interchanging the flows of processor-1 in S_1 with the flows of processor-2 in S_2 (i.e., interchanging flows at distance of more than $L/2$ from the origin through the $L/2$ symmetry-axis). This interchange will leave the congestion cost the same, yet obviously improve the overall connection cost, contradicting the optimality of the assignment. □

We next obtain the optimal cost for one of the users, and omit user indices for simplicity of exposition. Using (10), we have $u(x) = h^{-1}\big(\lambda - \alpha x\big)$. Noting that $h^{-1}(u) = \frac{u}{2c}$ (by using (23)),

$$u(x) = \frac{\lambda - \alpha x}{2c}. \tag{32}$$

Hence, we may write the total flow constraint as follows: $\int_0^{\frac{L}{2}} (\lambda - \alpha x) dx = 2rc$. The last equation allows us to obtain λ. Indeed, the equation is equivalent to $\frac{\lambda L}{2} - \frac{\alpha L^2}{8} = 2rc$, or $\lambda = \frac{4rc}{L} + \frac{\alpha L}{4}$. The user's optimal cost is given by $J(\mathbf{u}) = \int_0^{\frac{L}{2}} u(x)\,(\alpha x + cu(x))\,dx$. Noting (32), cost becomes

$$\begin{aligned} J(\mathbf{u}) &= \int_0^{\frac{L}{2}} \frac{\lambda - \alpha x}{2c}\left(\alpha x + \frac{\lambda - \alpha x}{2}\right) dx \\ &= \frac{1}{4c}\int_0^{\frac{L}{2}} \left(\lambda^2 - \alpha^2 x^2\right) dx \\ &= \frac{1}{4c}\left(\frac{\lambda^2 L}{2} - \frac{\alpha^2 L^3}{24}\right) \\ &= \frac{1}{4}\left(\frac{8r^2 c}{L} + \alpha r L + \frac{\alpha^2 L^3}{32c} - \frac{\alpha^2 L^3}{24c}\right). \end{aligned}$$

Hence, the total optimal cost is given by

$$JO = 2J(\mathbf{u}) = \frac{4r^2 c}{L} + \frac{\alpha r L}{2} - \frac{\alpha^2 L^3}{192c}. \tag{33}$$

Price of Anarchy. We are now ready to compute the price of anarchy (PoA), which is the worst possible performance ratio between the equilibrium cost and optimal cost. We normalize L to one. Using (30) and (33) and multiplying both equilibrium and optimal cost by c, the PoA is given by

$$\begin{aligned} PoA = \sup_{\alpha, r, c} & \frac{4r^2c^2 + \alpha rc - \frac{\alpha^2}{6}}{4r^2c^2 + \frac{\alpha rc}{2} - \frac{\alpha^2}{192}} \\ \text{s.t} \quad & rc > \frac{1}{2}\alpha, \end{aligned} \tag{34}$$

where the constraint above follows from being under heavy load conditions. Define $y = \frac{rc}{\alpha}$; then (34) is equivalent to

$$PoA = \sup_{y \geq 0.5} \frac{4y^2 + y - \frac{1}{6}}{4y^2 + \frac{y}{2} - \frac{1}{192}}. \tag{35}$$

The (unique) extremum point of (35) is obtained at $y = 0.7015$ with a value of 1.0818, which is higher than the values obtained at $y = 0.5$ and $y \rightarrow \infty$. Hence, the PoA is exactly 1.0818.

VII. Conclusion

The main contribution of this paper is to introduce a continuum approximation of the topology in the context of networks routing games, which arise, e.g., in load balancing scenarios. The approach has its sources in the very rich road-traffic literature. The noncooperative game in transportation models differs from our setting in various aspects: There are infinitely many non-atomic players, and unlike our setting, it may admit a convex potential function whose solution corresponds to the equilibrium point. Even without this property, we have demonstrated in this paper the potential of working with the continuum approximation.

The advantage of using this approximation in cases where the number of resources is large is threefold: (i) the complexity of the solution becomes independent of the number of resources (or links), (ii) explicit closed form expressions can be obtained for the equilibrium value and strategies. (iii) the continuum equilibrium is the limit of the discrete case as the number of resources goes to infinity, hence it is legitimate to use the approximation to obtain structural equilibrium properties (analogous to fluid limits in queuing networks).

The inefficiency result obtained in this paper is somewhat surprising, considering the fact that the equilibrium coincides with the social optimum when no connection costs are involved. Our initial intuition was that connection costs would not affect this property, as these can be regarded as natural incentives to use close-by resources and enforce efficiency. A challenging direction for future work is to examine whether this negative geometric-effect carries over to more than two processors and to more complex cost structures and network topologies.

At a higher level, the characterization of the equilibrium in our system, and in particular the study of its efficiency loss, is significant not only in the context of our model, but also in the broader framework of networking games. Most of the research that has investigated efficiency loss in networks considers *single-class* costs, meaning that all users obtain identical per-bit costs for shipping flow at a given link. This property is no longer valid in our case, due to heterogenous per-link connection costs. *Multi-class* models, as our model, may be more common at higher networking-levels, where users usually have diverse utilities and are thus much harder to analyze. Our explicit bound on the PoA was derived due to the continuum of links approximation. This result further emphasizes the potential of applying the approximation to additional multi-class network domains.

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Appendix A
Proof of Theorem 1

The idea behind the proof is to reduce our network to a network of (infinitely many) parallel links, and then extend the known uniqueness results for finitely-many parallel links to the infinite case. We abide with the terminology and notations that were used in [20], and provide the uniqueness proof for a general finite number of players.

Let $J_x^i(u^i(x),u(x)) = u^i(x)\big(f^i(x)+g_x(u(x))\big)$ denote user i's total cost of using link x. We denote $K_x^i = \frac{\partial J_x^i}{\partial u^i(x)}$. Note that $K_x^i = K_x^i(u^i(x),u(x))$ is a function of two arguments. Due to Assumption 2, it is readily seen that K_x^i is strictly increasing in both $u^i(x)$ and $u(x)$.

Let $\mathbf{u}$ and $\hat{\mathbf{u}}$ be two NEs. $\mathbf{u}$ (and $\hat{\mathbf{u}}$) satisfy the KKT conditions (4), which may be written as

$$\begin{aligned} K_x^i(u^i(x),u(x)) &\geq \lambda^i \quad \text{if } u^i(x)=0, && (36)\\ K_x^i(u^i(x),u(x)) &= \lambda^i \quad \text{if } u^i(x)>0. && (37)\end{aligned}$$

The first step is to establish that $u(x)=\hat{u}(x)$ a.s. for every $x\in[0,L]$. To this end, we prove that for each x and i, the following relations hold (a.s.):

$$\begin{aligned} \big\{\hat{\lambda}^i \leq \lambda^i, \hat{u}(x)\geq u(x)\big\} &\;\rightarrow\; \hat{u}^i(x)\leq u^i(x), && (38)\\ \big\{\hat{\lambda}^i \geq \lambda^i, \hat{u}(x)\leq u(x)\big\} &\;\rightarrow\; \hat{u}^i(x)\geq u^i(x). && (39)\end{aligned}$$

We shall only prove (38), since (39) is symmetric. Assume that $\hat{\lambda}^i \leq \lambda^i$ and $\hat{u}(x)\geq u(x)$ for some x and i. Note that (38) holds trivially if $\hat{u}^i(x)=0$, thus we only have to consider the case where $\hat{u}^i(x)>0$. In that case, the KKT conditions (36) along with our assumptions imply that

$$\begin{aligned} K_x^i(\hat{u}^i(x),\hat{u}(x)) = \hat{\lambda}^i \leq \lambda^i = K_x^i(u^i(x),u(x))\\ \leq K_x^i(u^i(x),\hat{u}(x)), \qquad (40)\end{aligned}$$

where the first equality and first inequality follow from the KKT conditions, and the last inequality follows from the monotonicity of K_x^i in its second argument. Now, since K_x^i is strictly increasing in its first argument, this implies that $\hat{u}^i(x)\leq u^i(x)$, and (38) is established.

Let $\mathcal{X}_1 = \big\{x : \hat{u}(x) > u(x)\big\}$, also denote $\mathcal{I}(x) = \big\{i : \hat{\lambda}^i > \lambda^i\big\}$, $\mathcal{X}_2 = \mathcal{X} - \mathcal{X}_1 = \big\{x : \hat{u}(x)\leq u(x)\big\}$. Assume that $\mathcal{X}_1$ is not empty. Recalling that $\int\limits_x \hat{u}^i(x) = \int\limits_x u^i(x) = r_i$, it follows from (39) that for every $i\in\mathcal{I}(x)$

$$\int\limits_{x\in\mathcal{X}_1} \hat{u}^i(x) = r_i - \int\limits_{x\in\mathcal{X}_2} \hat{u}^i(x) \leq r_i - \int\limits_{x\in\mathcal{X}_2} u^i(x) = \int\limits_{x\in\mathcal{X}_1} u^i(x). \qquad (41)$$

Noting that (38) implies that $\hat{u}^i(x)\leq u^i(x)$ for $a\in\mathcal{X}_1$ and $i\notin\mathcal{I}(x)$, it follows that

$$\int\limits_{x\in\mathcal{X}_1} \hat{u}(x) = \int\limits_{x\in\mathcal{X}_1}\int\limits_{i\in\mathcal{I}} \hat{u}^i(x) \leq \int\limits_{x\in\mathcal{X}_1}\int\limits_{i\in\mathcal{I}} u^i(x) = \int\limits_{x\in\mathcal{X}_1} u(x). \qquad (42)$$

This inequality contradicts the definition of $\mathcal{X}_1$, which implies that $\mathcal{X}_1$ is an empty set almost surely. By symmetry it may also be concluded that the set $\big\{x : \hat{u}(x) > u(x)\big\}$ is empty a.s. Thus, $\hat{u}(x) = u(x)$ a.s.

We now proceed to show that $\hat{u}^i(x) = u^i(x)$ a.s. To this end, note that (38) may be strengthened as follows:

$$\begin{aligned} &\big\{\hat{\lambda}^i < \lambda^i, \hat{u}(x) = u(x)\big\} \Rightarrow && (43)\\ &\hat{u}^i(x) < u^i(x) \quad \text{or} \quad \hat{u}^i(x) = u^i(x) = 0.\end{aligned}$$

Indeed, if $\hat{u}^i(x)=0$ the implication is trivial. Otherwise, if $\hat{u}^i(x)>0$ it follows similarly to (40) that $K_x^i(\hat{u}^i(x),\hat{u}(x)) < K_x^i(u^i(x),\hat{u}(x))$, so that $\hat{u}^i(x) < u^i(x)$ as required.

Assume now that $\hat{\lambda}^i < \lambda^i$ for some i. Since $\int\limits_{x\in\mathcal{X}} \hat{u}^i(x) = r_i$, then $\hat{u}^i(x) > 0$ for a set of links of measure greater than zero. (43) thus imply that $r_i = \int\limits_{x\in\mathcal{X}} \hat{u}^i(x) < \int\limits_{x\in\mathcal{X}} u^i(x)$, which contradicts the demand constraint for user i. A symmetrical contradiction may be obtained for the case where $\hat{\lambda}^i > \lambda^i$. Hence, $\hat{\lambda}^i = \lambda^i$. Recalling that $\hat{u}(x) = u(x)$ a.s. A straightforward use of (38), (39) implies that $\hat{u}^i(x) = u^i(x)$ a.s for every $x\in[0,L]$. $\square$

APPENDIX B
PROOF OF EXISTENCE

We establish below the existence of an equilibrium point by restricting users to a subset of policies. The restriction would be perceived natural in the context of the problem instances that we consider from Section IV onwards.

Assumption 3: Each user i chooses its strategy u^i from the subset of strategies S^i that are uniformly bounded and equi-continuous (see [17] p. 102) functions with respect to x.

Theorem 6: Let Assumptions 1–3 hold. Then there exists a Nash Equilibrium (3) for the noncooperative game.

Proof: We show that under Assumptions 1–3, the game is a convex game with infinite-dimensional action space [19]. Existence of an equilibrium point in such game follows from Theorem 3.1 in [19].

A) Consider the class of continuous functions $\mathcal{C}$ equipped with the topology induced by the $\sup$ norm. By Assumption 3, the set $S^i\subset\mathcal{C}$ is convex and also compact in $\mathcal{C}$ by the Arzela-Ascoli Theorem (see [17] p. 102). Hence the joint set of policies $S = S^1\times S^2$ is a convex and compact set.

Details:*The set $S^i\subset\mathcal{C}$ is convex*: Let $h^1(x), p^1(x)\in S^1$. We show that $\alpha h^1(x) + (1-\alpha)p^1(x)\in S^1$, $0\leq\alpha\leq 1$. To prove that, we need to show that both the uniformly-bounded and equi-continuity properties are preserved.

1) *Uniformly-bounded*: Assume that $h^1(x) < K$ and $p^1(x) < K$ for some $K>0$ and every $x\in[0,L]$. Thus,

$$\begin{aligned}&\alpha h^1(x) + (1-\alpha)p^1(x) \leq \alpha\sup_x h^1(x) + (1-\alpha)\sup_x p^1(x)\\ &\leq \alpha K + (1-\alpha)K = K.\end{aligned}$$

2) *Equi-continuity.* Since $h^1, p^1\in S^1$, for any $\epsilon>0$ there exists a number $\delta>0$ such that $|x_1-x_2|<\delta$ implies that $|h^1(x_1)-h^1(x_2)|<\epsilon$ and $|p^1(x_1)-p^1(x_2)|<\epsilon$. We show below that a convex combinations of the two functions obeys

the same property. Indeed,

$$\begin{aligned}&\left|\alpha h^1(x_1)+(1-\alpha)p^1(x_1)-\alpha h^1(x_2)-(1-\alpha)p^1(x_2)\right|\\&=\left|\alpha[h^1(x_1)-h^1(x_2)]+(1-\alpha)[p^1(x_1)-p^1(x_2)]\right|\\&\le \alpha\epsilon+(1-\alpha)\epsilon=\epsilon,\end{aligned}$$

where the inequality above follows by the triangle inequality.

B) User i's cost J^i is convex with respect to its own strategy $u^i \in S^i$. This property follows by the point-wise convexity in $u^i(x)$ and the separability of the cost function in x. Indeed, J^i is clearly convex in $u^i(x)$ by the convexity assumption on $g_x(u(x))$.

Details:
1) point-wise convexity: $\frac{\partial J^i(u)}{\partial u^i(x)} = f^i(x) + g_x(u(x)) + u^i(x)g'_x(u(x))$. The first summand is not a function of $u(x)$; the second summand increases with $u(x)$ by assumption; so does the third summand, Since $g(u(x))$ is convex increasing in $u(x)$.
2) Convexity: Fixing u^2, we express $J^1(u) \equiv \tilde{J}(u^1)$ as $\int_x \tilde{J}^1(u^1(x))dx$. Then

$$\begin{aligned}&\tilde{J}^1\big(\alpha h^1+(1-\alpha)p^1\big)=\int_x \tilde{J}^1(\alpha h^1(x)+(1-\alpha)p^1(x))dx\\&\le \int_x \alpha\tilde{J}^1(h^1(x))+(1-\alpha)\tilde{J}^1(p^1(x))dx\\&=\alpha\tilde{J}^1(h^1)+(1-\alpha)J^1(p^1),\end{aligned}$$

where the inequality above follows by the point-wise convexity shown above.

C) The sum of payoffs $J^1 + J^2$ is obviously continuous over the strategy space S (by the continuity of the functions g and f).

D) Similarly, J^i is continuous in $u^j \in S^j$.

Due to properties A–D, our game is a convex game, for which an equilibrium exists [19]. □

APPENDIX C
CONVERGENCE OF THE FINITE-GAME TO THE CONTINUUM-GAME

We show in this section the convergence of values and equilibria the game with finite number of links to that with the continuum of links as the number of link grows to infinity. Let N be the number of links in the initial finite game. The cost function associated with link m is g_m. The connection cost for player i associated with a link at point x is $f^i(x)$.

We consider a sequence of games where the nth game has the following characteristics:

- We replace each link k of the original model with a group of n links identical to each other and equally spaced; we refer to this group as the kth cluster.
- The delay cost associated with link m belonging to cluster k where $n(k-1)+1 \le m \le nk$ $(k = 1, ..., N)$ is $g^n_m(u) = g_k(nu)$.
- The connection cost for player i associated with a link at point x is $f^i(x)$.
- Let $\mathbf{U}^n_i$ denote the set of strategies for player i.
- The demands r_i do not depend on n. Thus for any $\mathbf{u}^{n,i} \in \mathbf{U}^n_i$, $\sum_{m=1}^{nN} u^{n,i}_m = r_i$.

The *continuous game* is assumed to have the same connection costs $f^i(x)$ as the initial game. We define the interval $X(m) := [(m-1)/L, m/L)$ to be the mth cluster in the continuum model. The delay cost density g^c_x at any point x in cluster m is taken to be $g^c_x(u) = Ng_m(u)/L$ where $m \in \{1, ..., N\}$. Let $\mathbf{U}_i$ be the set of strategies for player i in the continuous game.

Remark 3: The choice of the delay costs g^n_m (in the nth game) and of g^c_x in the limit game are done so that the total delay cost related to cluster k is unchanged.
Indeed, if an amount u^i_k of flow is sent by player i in the initial finite model over link k, and if in the nth game this player splits that flow equally among the corresponding n links in cluster k then

$$u^{n,i}_m g^n_m(u^{n,1}_m + u^{n,2}_m) = \frac{u^i_m}{n} g_m(u^1_m + u^2_m)$$

so that the total delay cost related to cluster k is unchanged.
Similarly, if in the continuous game the amount u^i_k of flow is split equally over the kth cluster then

$$\int_{X(k)} u^i(x)g_x(u^1(x)+u^2(x))dx =$$

$$\frac{L}{N}u^i_m g_x(u^1(x)+u^2(x)) = u^i_m g_m(u^1_m+u^2_m)$$

yielding the same conclusion.

Theorem 7: There is some constant B such that for any strategy of player i, any best response for player $j \neq i$ satisfies $u^j(x) \le B$ almost everywhere for all x for the limit game.

Proof: Define

$$F := \max(f^1(L), f^2(0)), \quad q := \left.\frac{\mathrm{d}g_x(u)}{\mathrm{d}u}\right|_{u=(r^1+r^2)/L}$$

Let

$$G = F + \max_x g_x(r^i/L + r^j/L)$$

Since $g_x(u)$ is piecewise constant in x and convex increasing in u, there exists B be such that

$$\min_x(g_x(B - r_i/L) - g_x(0)) \ge G.$$

Fix any strategy for player i. Assume that $u^j(x) > B$ over a set S of measure $\alpha > 0$.

Consider the "water-filling" strategy v^j whose density is given by $v^j(x) = \min(\beta - u^i(x) - u^j(x), 0)$ where β is the unique constant for which $\int_0^L v^j(x)dx = \int_S (u^j(x) = r_j/L)dx$ (and hence $\int_0^L v^j(x)dx = \int_S u^j(x)dx - \alpha r_j/L$.

Consider the strategy $\overline{u}^j$ that is obtained from u^j as follows. At $x \in S$ it sends r_j/L. For $x \notin S$ it is given by $\overline{u}^j(x) = u^j(x) + v^j(x)$.

Clearly $\beta \le (r^i + r^j)/L$. Hence

$$J^j(\overline{\mathbf{u}}^i, \mathbf{u}^j) \le J^j(\mathbf{u}^i, \mathbf{u}^j) + \alpha G$$

$$+ \int_S \big(g_x(u^i(x) + r^j/L) - g_x(u^i(x) + u^j(x))\big)dx$$

Due to the convexity of g_x, the last integral is smaller than $-\int_S \big(g_x(u^j(x) - r^j/L) - g_x(0)\big)dx$ which is smaller than $-\alpha G$. We conclude that $J^j(\overline{\mathbf{u}}^i, \mathbf{u}^j) \leq J^j(\mathbf{u}^i, \mathbf{u}^j)$ so that $\mathbf{u}^j$ cannot be a best response to $\mathbf{u}^i$. $\square$

Similarly one can prove the following:

Theorem 8: There is some constant B such that for any strategy of player i, any best response for player $j \neq i$ satisfies $u_n^j(x) \leq B/n$ for all x for the nth game for all n.

In view of the above Theorems, we can restrict our search for equilibria to strategies that are uniformly bounded in the sense of the above theorems. With an abuse of notation we shall denote the class of bounded strategies for the nth and the limit games with the same notation that we already used for the sets of all strategies.

Definition C.1: Consider a strategy pair $(\mathbf{u}^{n,1}, \mathbf{u}^{n,2})$ for the nth game. Define $\pi_i^n(\mathbf{u}^{n,i})$ to be the strategy for the continuous game obtained from $\mathbf{u}^{n,i}$ by setting

$$\pi_i^n(\mathbf{u}^{n,i})(x) = \mathbf{u}_m^{n,i} \times \frac{nN}{L}$$

for all x such that $(m-1)L \leq x < mL$, $m = 1, ..., nN$.

Definition C.2: Consider a strategy pair $(\mathbf{u}^1, \mathbf{u}^2)$ for the continuum game. Define $\sigma_i^n(\mathbf{u}^i)$ to be the strategy for the nth game obtained from $\mathbf{u}^i$ by setting

$$\Big(\sigma_i^n(\mathbf{u}^i)\Big)_m = \int_{(m-1)L/nN}^{mL/nN} u_m^i(x)\mathrm{d}x, \quad m = 1, ..., nN$$

We shall assume throughout that f and g are such that the following 3 assumptions hold for $i = 1, 2$, $j = 3 - i$:

- **A1:** J^i is continuous in $\mathbf{U}_i$ uniformly over $\mathbf{U}_j$ for some topology on $\mathbf{U}_i$, $i = 1, 2$.
- **A2:** $\lim_{n\to\infty} |J^i(\sigma_1^n(\mathbf{u}^1), \sigma_2^n(\mathbf{u}^2)) - J^i(\mathbf{u}^1, \mathbf{u}^2)| = 0$ uniformly over $\mathbf{U}_1, \mathbf{U}_2$;
- **A3:** $\lim_{n\to\infty} |J^{i,n}(\sigma_i^n(\mathbf{v}^i), \mathbf{u}^{j,n}) - J^i(\mathbf{v}^i, \pi_j^n(\mathbf{u}^{j,n}))| = 0$ for any sequence $\mathbf{u}^{j,n} \in \mathbf{U}_j^n$ and $\mathbf{v}^i \in \mathbf{U}_i$; the convergence is moreover *uniform* over all sequences $\mathbf{u}^{j,n}$ and all $\mathbf{v}^i \in \mathbf{U}_i$.

Remark 4: (i) As example for Assumption A1, we may consider $\mathbf{U}_i$ to be equipped with the topology induced by the L_∞ norm; J^i is then continuous in $\mathbf{u}^i$ uniformly in $\mathbf{u}^j$ if for any δ there is some $\epsilon > 0$ such that for any $\mathbf{u}^i, \mathbf{v}^i$ satisfying $\sup_{0\leq x\leq L} |u^i(x) - v^i(x)| < \epsilon$ and for any $\mathbf{u}^j$, we have

$$|J^i(\mathbf{u}^i, \mathbf{u}^j) - J^i(\mathbf{v}^i, \mathbf{u}^j)| < \delta \tag{44}$$

(ii) Assumptions A2 and A3 state that for fixed strategies, the costs converge as $n \to \infty$ to those of the limit game uniformly.

We state some of many convergence results that hold. The proof follows from [5, Thm 3.1].

Theorem 9: Under Assumptions A1-A3,

(i) Let $(\mathbf{u}^{i,n}, i = 1, 2)$ be an equilibrium for the nth game. Any accumulation point $(\mathbf{u}^i \in \mathbf{U}_i, i = 1, 2)$ of the sequence of strategy pairs $(\pi_i^n(\mathbf{u}^{i,n}), i = 1, 2)$ is an equilibrium for the limit game.

(ii) Let $(\mathbf{u}^i, i = 1, 2)$ be an equilibrium for the limit game. Then for any $\epsilon > 0$, $(\sigma_i^n(\mathbf{u}^i), i = 1, 2)$ is an ϵ-equilibrium for all n sufficiently large.