Decentralized Team Decision Via Coding

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***Abstract*— We consider a class of decentralized team decision problems with non classical information structure and discrete state spaces. In general, decentralized team decision problems are NP-complete. In this paper, we present a simple class of problems where an optimal solution can be obtained via coding. This class is motivated by a famous mathematical puzzle called the hats problem.**

I. Extended Abstract

In this paper, we consider a simple class of decentralized team decision problems over discrete state spaces with non classical information structure. Tsitsiklis and Athans [1] showed that in general, decentralized team decision problems are NP-complete. We show that for a class of problems presented here, an optimal solution can be derived via coding. This class of problems is motivated by a famous mathematical puzzle called the hats problem [2], [3].

A classic variant of the hats problem is as follows. There are 10 prisoners and each prisoner is randomly assigned a hat, either red or blue in color. The prisoners are lined up in a single file where each prisoner can see the color of hats in front of him but not behind him. The objective is to devise a strategy so that each prisoner says either red or blue. If at least 9 prisoners correctly say the color of their own hat, then all of them are let free, else they all die. Note that once the strategy is devised, the prisoners are not allowed to communicate and the strategy should work for every instance of hat assignment. The solution lies in a clever use of coding. Let us assume that the red hat is denoted by 0 and the blue hat is denoted by 1. Let us also number the prisoners such that the prisoner at the back of the line is number 1. The first prisoner then sees the hats of all prisoners except himself. He computes the binary sum of the hats of other prisoners and says the sum instead of the color of his own hat. The subsequent prisoners can now correctly infer their own hat. To do so, they compute the binary sum of the hats in front of them and add it to the color of the hats of all prisoners behind them (except prisoner 1). Comparing this to the binary sum given by the first prisoner they can correctly compute their own hat.

This classic hats problem can be viewed as a decentralized team decision problem. We index the players in the team decision problem by $i = 1, \cdots, N$. Each player is assigned a state $h_i \in \{0, 1\}$, where we assume that the state are independently and identically distributed with $\mathrm{Prob}(h_i = 1) = p$ and $\mathrm{Prob}(h_i = 0) = q$. We also assume (without loss of generality) that $p \geq 1/2$. Let g_i denote the action taken by player i. Let $\mathcal{I}_i$ be the information available to player i. Then, we have

$$\mathcal{I}_i = \{h_{i+1}, h_{i+2}, \cdots h_N, g_1, g_2, \cdots, g_{i-1}\}.$$

The objective is for each player to choose its action g_i to minimize the cost function

$$\begin{aligned} J &= \frac{1}{N}\mathbb{E}\big[\sum_{i=1}^{N} \mathbf{1}_{g_i \neq h_i}\big] \\ &= \frac{1}{N}\sum_{i=1}^{N} \mathrm{Prob}(g_i \neq h_i). \end{aligned}$$

Here J is the average number of incorrect guesses. It is evident that the information structure is not partially-nested as defined in [4]. However, we show that an optimal policy for this team decision problem can be found via coding. In particular, we show that it is optimal for the first player to use its action g_1 to signal some information to other players about their states $h_2, \cdots, h_N$. This signaling is done by encoding the states of other players in its action as $g_1 = h_2 + h_3 + \cdots h_N + \mathbf{1}_{\text{N-odd}}$, where this sum is interpreted as a binary sum and $\mathbf{1}_{\text{N-odd}}$ is the indicator function that the total number of players is odd. Using this action, every other player can correctly estimate its own state. We show that this particular form of coding scheme is optimal. This problem illustrates a connection between team decision and signaling which was alluded to in [5].

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