Kalman Filtering
with
Partial Observation Losses

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Motivation

Figure 1: Distributed Control Systems

- Networked control systems.
- Communication links can be unreliable. Packets can be lost or exceedingly delayed.
The optimal controller for Linear Quadratic Gaussian (LQG) control is Kalman Filter + State feedback controller.

- The Kalman filter requires multiple packets to form the complete observation of the control state.

Figure 2: System Block Diagram
Previous Work

• R. E. Kalman - Classical Kalman filter assumes periodic measurement updates.
  – The error covariance matrix iterates according to an ARE:
    \[ P_{t+1} = A P_t A' + Q - A P_t C' (C P_t C' + R)^{-1} C P_t A'. \]

• Sinopoli et. al. [CDC 2003] first addressed Kalman filtering with observation losses.
  – The observation is either received in full or lost completely.
  – The error covariance matrix iterates stochastically:
    \[ P_{t+1} = A P_t A' + Q - \gamma_t A P_t C' (C P_t C' + R)^{-1} C P_t A'. \]
Problem Formulation

- System Dynamics:

\[
\begin{align*}
x_{t+1} &= Ax_t + w_t, \\
\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x_t + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix},
\end{align*}
\]

- \( \gamma_{i,t} \) indicates whether \( y_{i,t} \) is received correctly in time step \( t \).
- \( \gamma_{1,t} \) and \( \gamma_{2,t} \) are i.i.d. Bernoulli random variables for all \( t \).
- \( \Pr(\gamma_{1,t} = 1) = \lambda_1 \) and \( \Pr(\gamma_{2,t} = 1) = \lambda_2 \).
- The measurement noise is conditional on packet arrivals:

\[
p(v_{i,t}|\gamma_{i,t}) \sim \begin{cases} 
\mathcal{N}(0, R_{ii}) & \text{if } \gamma_{i,t} = 1, \\
\mathcal{N}(0, \sigma_i^2 I) & \text{if } \gamma_{i,t} = 0,
\end{cases}
\]

where \( \sigma_i \to \infty \).
Kalman Filter updates in the presence of random packet losses

- When both measurements are received, same as classical Kalman filter.
- When both measurements are lost, the Kalman filter goes forward one step open loop.
- When only one measurement is received, the Kalman filter updates as if the observation only consists of one measurement in the classical Kalman filter.
- The iteration is stochastic.

\[
P_{t+1} = AP_tA' + Q - \gamma_{1,t}\gamma_{2,t}AP_tC''(CP_tC' + R)^{-1}CP_tA' \\
- \gamma_{1,t}(1 - \gamma_{2,t})AP_tC'_1(C_1P_tC_1' + R_{11})^{-1}C_1P_tA' \\
-(1 - \gamma_{1,t})\gamma_{2,t}AP_tC'_2(C_2P_tC_2' + R_{22})^{-1}C_2P_tA'.
\]
**Function** $g_{\lambda_1 \lambda_2}(X)$

- Let

$$g_{\lambda_1 \lambda_2}(X) = AXA' + Q - \lambda_1 \lambda_2 AXC'(CXC' + R)^{-1}CXA'$$
  $$-\lambda_1 (1 - \lambda_2) AXC_1'(C_1XC_1' + R_{11})^{-1}C_1XA'$$
  $$-(1 - \lambda_1) \lambda_2 AXC_2'(C_2XC_2' + R_{22})^{-1}C_2XA'.$$

- Then

$$E[P_{t+1}|P_t] = g_{\lambda_1 \lambda_2}(P_t),$$

and

$$E[P_{t+1}] = E[g_{\lambda_1 \lambda_2}(P_t)].$$

- Note that $P_t$ is bounded with probability 1 if $E[P_t]$ is bounded.
An Auxiliary Function

- Let \( \phi(K, K_1, K_2, X) = (1 - \lambda_1)(1 - \lambda_2)(AXA' + Q) + \lambda_1\lambda_2(FF' + V) + \lambda_1(1 - \lambda_2)(F_1F_1' + V_1) + \lambda_2(1 - \lambda_1)(F_2F_2' + V_2), \)

where \( F = A + KC, \) \( F_1 = A + K_1C_1, \) \( F_2 = A + K_2C_2, \)
\( V = Q + KRK', \) \( V_1 = Q + K_1R_{11}K_1', \) \( V_2 = Q + K_2R_{22}K_2' \) and \( X \geq 0. \)

- \( \exists K_x, K_{x1}, K_{x2}, \) such that \( g_{\lambda_1\lambda_2}(X) = \phi(K_x, K_{x1}, K_{x2}, X). \)

- \( g_{\lambda_1\lambda_2}(X) = \min_{K, K_1, K_2} \phi(K, K_1, K_2, X). \)
Statistical Properties of $g_{\lambda_1 \lambda_2}(X)$

- $g_{\lambda_1 \lambda_2}(X)$ is a concave function in $X$ for $X \geq 0$.
- $g_{\lambda_1 \lambda_2}(X)$ is a nondecreasing function in $X$.
- If $0 \leq \lambda_1 \leq 1$ is fixed and $0 \leq \lambda_2^{(1)} \leq \lambda_2^{(2)} \leq 1$, then $g_{\lambda_1 \lambda_2^{(1)}}(X) \geq g_{\lambda_1 \lambda_2^{(2)}}(X)$. Similarly, if $0 \leq \lambda_2 \leq 1$ is fixed and $0 \leq \lambda_1^{(1)} \leq \lambda_1^{(2)} \leq 1$, then $g_{\lambda_1^{(1)} \lambda_2}(X) \geq g_{\lambda_1^{(2)} \lambda_2}(X)$.
- If $X \geq 0$ is a random variable, then

\[(1 - \lambda_1)(1 - \lambda_2)AE[X]A' + Q \leq E[g_{\lambda_1 \lambda_2}(X)] \leq g_{\lambda_1 \lambda_2}(E[X]).\]
Convergence Properties

• Under certain conditions, the iteration \( \bar{P}_{t+1} = g_{\lambda_1 \lambda_2} (\bar{P}_t) \) converges to a unique positive definite matrix.

• We show the existence of a sharp transition curve in the rectangular region \( 0 \leq \lambda_1 \leq 1 \) and \( 0 \leq \lambda_2 \leq 1 \) such that the convergence of the iteration \( \bar{P}_{t+1} = g_{\lambda_1 \lambda_2} (\bar{P}_t) \) changes.

• Both an upper bound and a lower bound for this transition curve can be computed.

• Both an upper bound and a lower bound for \( \lim_{t \to \infty} E[P_t] \) can be computed.
The Upper and Lower bounds of critical rates

**Theorem 1** For a given $\lambda_1$, we have an upper bound and a lower bound for $\lambda_2c$, i.e., $\lambda_2c \leq \lambda_2 \leq \overline{\lambda}_2c$, and

$$\lambda_2c = \arg \inf_{\lambda_2} [\exists \hat{S} > 0 | \hat{S} = (1 - \lambda_1)(1 - \lambda_2)A\hat{S}A' + Q]$$
$$= \max[1 - \frac{1}{\alpha^2(1 - \lambda_1)}, 0],$$

where $\alpha = \max_i |\sigma_i|$ and $\sigma_i$ is the $i^{th}$ eigenvalue of $A$;

$$\overline{\lambda}_2c = \arg \inf_{\lambda_2} [\exists \hat{X} | \hat{X} > g_{\lambda_1\lambda_2}(\hat{X})]$$
$$= \arg \inf_{\lambda_2} [\exists \hat{K}, \hat{K}_1, \hat{K}_2, \hat{X} > 0 | \hat{X} > \phi(\hat{K}, \hat{K}_1, \hat{K}_2, \hat{X})].$$

Again, the result holds similarly if $\lambda_2$ is fixed.
Computing $\bar{\lambda}_{2c}$

**Theorem 2** If $(A, Q)$ is controllable, then the following three statements are equivalent:

(a) $\exists \bar{X}$ such that $\bar{X} > g_{\lambda_1 \lambda_2}(\bar{X})$.

(b) $\exists \bar{K}, \bar{K}_1, \bar{K}_2, \bar{X} > 0$ such that $\bar{X} > \phi(\bar{K}, \bar{K}_1, \bar{K}_2, \bar{X})$.

(c) $\exists \bar{Z}, \bar{Z}_1, \bar{Z}_2$ and $0 < \bar{Y} \leq I$ such that $\Psi(Y, Z, Z_1, Z_2) > 0$, where

$$\Psi(Y, Z, Z_1, Z_2) =$$

$$\begin{bmatrix}
Y & \sqrt{\lambda_1 \lambda_2}(YA + ZC) & \cdots & \cdots & \cdots \\
\sqrt{\lambda_1 \lambda_2}(A'Y + C'Z') & Y & 0 & 0 & 0 \\
\sqrt{\lambda_2(1 - \lambda_1)}(A'Y + C'_2Z'_2) & 0 & Y & 0 & 0 \\
\sqrt{\lambda_1(1 - \lambda_2)}(A'Y + C'_1Z'_1) & 0 & 0 & Y & 0 \\
\sqrt{(1 - \lambda_1)(1 - \lambda_2)}A'Y & 0 & 0 & 0 & Y
\end{bmatrix}.$$
Computing the Upper and Lower bounds of the expected Error Covariance Matrix

- Assume \((A, Q)\) is controllable, \((A, C)\) is observable and \(A\) is unstable. For a fixed \(\lambda_1\), if \(\lambda_2 > \lambda_{2c}\), then we can find positive semidefinite matrices \(\bar{S} \geq 0\) and \(\bar{V} \geq 0\) such that

\[
0 \leq \bar{S} \leq \lim_{t \to \infty} E[P_t] \leq \bar{V}, \quad \forall E[P_0] \geq 0,
\]

where \(\bar{S} = (1 - \lambda_1)(1 - \lambda_2)A\bar{S}A' + Q\) and \(\bar{V} = g_{\lambda_1 \lambda_2}(\bar{V})\).

- \(\bar{V}\) can also be found via solving the following SDP:

\[
\begin{align*}
\begin{cases}
\arg \max_V & \text{Trace}(V) \\
\text{subject to} & \Gamma(V) \geq 0.
\end{cases}
\end{align*}
\]
Numerical Example 1 - When the two bounds meet

- \( A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix} \), \( C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \), \( C = [C_1; C_2] \).

- We assume the noise covariance matrix \( Q = 20I_2 \) and \( R = 2.5I_4 \).

- The upper bound and lower bound coincide and \( \lambda_{2c} = \max[0, 1 - \frac{1}{\alpha^2(1-\lambda_1)}] \), where \( \alpha = \max_i |\sigma_i| \) and \( \sigma_i \) is the \( i^{th} \) eigenvalue of matrix \( A \).
Figure 3: Upper bound and Lower bound of the $\lim_{t \to \infty} E[P_t]$. 
Figure 4: Upper bound and Lower bound of the $\lim_{t \to \infty} E[P_t]$. 
Numerical Example 2 - When the two bounds do not meet

• In general, the upper bound and the lower bound are not equal.

• \[ A = \begin{bmatrix} 2.5 & 0 \\ 0 & 1.5 \end{bmatrix} \] and \( C = I_2 \). Then \( C_1 = [1 \ 0] \) and \( C_2 = [0 \ 1] \).

• We assume the noise covariance matrix \( Q = 20I_2 \) and \( R = 2.5I_2 \).

• The upper bound is tight.
Figure 5: Upper and Lower bounds of the Stable Throughput Region
Network Resource Allocation

• Total network throughput is limited.
• Allocate the total throughput to minimize the upper bound.
• We need to solve the following SDP.

\[
\begin{align*}
\min_{\lambda_1, \lambda_2} & \quad \max_V \quad \text{Trace}(V) \\
\text{subject to} & \quad \Gamma(\lambda_1, \lambda_2, V) \geq 0, \\
& \quad \lambda_1 + \lambda_2 \leq t, \\
& \quad 0 \leq \lambda_1 \leq 1, \\
& \quad 0 \leq \lambda_2 \leq 1,
\end{align*}
\]
Numerical Example 3

- \( A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix} \), \( C_1 = [1 \ 0] \), \( C_2 = [1 \ 1] \), \( C = [C_1; C_2] \), 

\( Q = 20I_2 \) and \( R = 2.5I_2 \).

Figure 6: Upper and Lower bounds of the Stable Throughput Region
Optimal Allocation

Figure 7: Optimal Throughput Allocation
Conclusions and Future Work

• Kalman Filtering with partial observation losses has a stochastic error covariance matrix iteration.

• We find an upper bound and a lower bound of $\lim_{t \to \infty} \mathbb{E}[P_t]$.

• We find a throughput region that guarantees bounded error covariance matrix and a throughput region that leads to unbounded error.

• Kalman filtering with bursty packet losses.