Achievable rates with amplify-and-forward (AF) and decode-and-forward (DF) cooperative strategies are examined for relay networks. Motivated by sensor network applications, power-constrained networks with large bandwidth resources and a large number of nodes are considered. It is shown that AF strategies do not necessarily benefit from the available bandwidth. Rather, transmitting in the optimum AF bandwidth allows the network to operate in the linear regime where the achieved rate increases linearly with the available network power. The optimum power allocation among the AF relays, shown to be a form of maximal ratio combining, indicates the favorable relay positions. Orthogonal node transmissions are also examined. While the same optimum bandwidth result still holds, the relay power allocation in this case can be viewed as a form of water-filling. In contrast, the DF strategy will optimally operate in the wideband regime and is shown to require a different choice of relays. Thus, in a large scale network, the choice of a coding strategy goes beyond determining a coding scheme at a node; it also determines the operating bandwidth, as well as the set of relays and best distribution of the relay power.

Index Terms—Antenna arrays, optimum relay powers, relay channels, multi-hop cooperative strategies.

I. INTRODUCTION

Although the capacity of the single-relay channel [1] and consequently of wireless relay networks remains unknown, several coding strategies have been proposed. Two coding strategies were developed in [2]. The first uses block Markov superposition encoding, random partitioning, and successive decoding and achieves the capacity of the degraded relay channel [2]. Two modifications that somewhat simplify the above scheme were proposed in [3] and [4]. To avoid the random partitioning, backward decoding [3] and windowed decoding [4] were used, instead. The approach of [4] was extended to the general relay channel model and referred to as multihopping in [5]. When relays are close to the source, this strategy achieves the capacity for some wireless relay network models [6]. These are all decode-and-forward (DF) [7] strategies that require a relay to decode reliably the source message before forwarding.

A different paradigm in which relays do not decode the message, but send the compressed received values to the destination was proposed in [2] and extended to the multiple relay channel in [8]. When the relays are close to the destination, this strategy achieves the antenna-clustering capacity [8].

In another strategy that does not require reliable decoding at the relays, called amplify-and-forward (AF) [7], a relay forwards the scaled version of the received noisy copy of the source signal. Hence, the data is sent through only two- hops with no cooperation among relays. Under the assumption of uncoded transmission at the source, it was shown that the two-hop AF strategy achieves the asymptotic capacity in a relay network in the limit as the number of relays becomes large [9], [10]. It was further shown that in a random network the power efficiency of such a strategy increases with the number of relays [11]. Even though the relay power in [9]–[11] was allocated suboptimally, the favorable scaling was achieved due to the coherent combining of the relay signals that increases the received SNR at the destination node.

In this paper, we revisit the reliable, DF strategy and the unreliable, AF strategy used in a Gaussian network with many relays and a single source-destination pair. We show that these two strategies require two different modes of operation employing different sets of relays and different bandwidth and power allocation among them. Motivated by sensor networks, we consider networks in which transmit power is a limiting resource. Compared to the power, the bandwidth in such a network is plentiful. Operating in the wideband regime then seems like the right choice; at the expense of using a large number of degrees of freedom, the transmit energy per bit can be reduced.

However, the optimal operating regime for relay networks is in general unknown [12]. In this paper, we show that the AF strategy cannot necessarily benefit from the large bandwidth, i.e., that it should not operate in the wideband regime. The reason is that in the AF strategy, a part of the relay power is wasted amplifying the receiver noise. As the signaling bandwidth increases, the receiver noise increases and the AF gain reduces. Ultimately, for a network operating in the wideband regime, there is no benefit from relays employing the AF strategy [13].

In this paper, we characterize the optimum AF bandwidth and show that transmitting in the optimum bandwidth allows the network to operate in a linear regime where the achieved rate increases linearly with transmit power. Therefore, AF optimally uses only a fraction of the available dimensions. This same conclusion was recently shown in [14] for the single relay fading channel, when considering the outage capacity. We then...
We adopt a discrete-time Gaussian channel model [20] and let the vector $\mathbf{X}[n] = [X_0[n], \ldots, X_M[n]]^T$ denote the channel inputs in time slot $n$. The input $X_0[n]$ depends on the source message and the input $X_m[n]$ at relay $m$ depends on its past outputs $X_m[n] = f_m(Y_m[1], \ldots, Y_m[n-1])$.

In such a network, we consider two-hop forwarding strategies in which relays use only the information received from the source to choose their channel inputs and forward the messages to the destination. In the first hop, the source transmits. The channel output at relay $m$ is

$$Y_{Rm}[n] = \sqrt{\alpha_m}X_0[n] + Z_{Rm}[n]$$  \hspace{1cm} (1)

and at the destination

$$Y^{(1)}[n] = \sqrt{\beta_0}X_0[n] + Z[n]$$  \hspace{1cm} (2)

where $\sqrt{\alpha_m}$ and $\sqrt{\beta_0}$ are the source-relay $m$ and source-destination channel gain, respectively, and $Z[n]$ is a zero-mean Gaussian random variable with variance $N_0/2$. In the second hop, the relays transmit. In shared bandwidth, the channel output at the destination is

$$Y^{(2)}[n] = \sum_{m=1}^{M} \sqrt{\beta_m}X_m[n] + Z[n].$$  \hspace{1cm} (3)

Coefficients $\sqrt{\alpha_m}$ and $\sqrt{\beta_0}$ are assumed to be real numbers, for simplicity. When relays use orthogonal channels

$$Y^{(2)}[n] = \mathbf{B}\mathbf{X}[n] + Z[n]$$  \hspace{1cm} (4)

where $\mathbf{B} = \text{diag}(\sqrt{\beta_0}, \ldots, \sqrt{\beta_M})$ and $X_0[n] = 0$ and $\mathbf{Z}$ is a Gaussian noise vector with covariance matrix $\mathbf{K} = \sigma^2\mathbf{I}_{M+1}$. Using the cut-set Theorem [20, Th. 14.10.1], it was shown [9] that the capacity of this network is upper bounded by $\log M$, given that there is a dead zone around the source that contains no relays.

We consider two transmission strategies at the relays. As in [9], we consider the AF protocol at the relays, in which the noisy version of the source input $X_0$ received at relay $m$, $1 \leq m \leq M$ is amplified with gain $b_m \geq 0$ and forwarded with a unit delay. In time slot $n$, relay $m$ transmits

$$X_m[n] = \sqrt{b_m}(\sqrt{\alpha_m}X_0[n-1] + Z_{Rm}[n-1]).$$  \hspace{1cm} (5)

We also consider the DF strategy in which the source transmission is reliably received at a relay. The relay decodes, reencodes using an independent codebook and transmits.

In this paper, rather than considering the power constraint imposed on each transmitter, we assume that the total power budget of $p$ [Watts] is allocated to the network. The constraint is on the total power rather than on the power per dimension because DF and AF will not in general operate in the same bandwidth. Thus the power allocation over the relay nodes must satisfy the total power constraint

$$E[\mathbf{X}^{(i)T}\mathbf{X}^{(i)}] \leq p/2W^{(i)}, \quad i = 1, 2.$$  \hspace{1cm} (6)

II. SYSTEM MODEL

We consider a wireless Gaussian network with a single source, labeled node 0, the destination, labeled node $M + 1$, and $M$ relays that dedicate their resources to relaying information for the source. We consider two bandwidth allocations in the given network.

1) Shared bandwidth. All the relays transmit in a common bandwidth $W^{(1)}$.

2) Orthogonal channels. Every node is assigned to transmit in an orthogonal channel of bandwidth $W^{(2)}$. 

present the optimum power allocation among the AF relays. The solution can be viewed as a form of maximum ratio combining (MRC) with the powers being proportional to the equivalent channel gains that depend not only on the relay-destination channel gains, but also on the source-relay links.

The requirement of coherent combining of the signals transmitted from the different nodes may be too demanding in practice. Furthermore, a network with a large bandwidth available, can support orthogonal signaling at the nodes that precludes coherent combining at a receiver. To evaluate the performance of a two-hop network that does not benefit from coherent combining, we also consider a relay network model with orthogonal transmissions at the nodes. We show that the above result for the optimum bandwidth applies to this channel model as well. Furthermore, for the case of AF orthogonal transmissions, we again identify the best subset of AF relay nodes. The optimum relay power solution in this case can be viewed as a form of water-filling.

The optimum AF bandwidth and relay powers can be contrasted to the DF solution. In a network with unconstrained bandwidth, the DF strategy will operate in the wideband regime to minimize the energy cost per information bit [15], [16]. The wideband DF strategy requires again a different choice of relay nodes; for the orthogonal transmission case, we present the optimum solution and demonstrate that, the data should be sent through one DF relay that has the best pair of channels, when channels are determined by attenuation [17].

The results presented in this paper indicate that in a large scale network, a choice of a coding strategy goes beyond determining a coding scheme at a node; it also determines the operating bandwidth as well as the best distribution of the relay power. While we consider a single source-destination pair, our results have implications to networks with multiple source-destination pairs. Our view is that, for each such pair, the relay network in between is a resource that we aim to use efficiently. Such a view motivates a total power constraint as the network budget. The optimum power allocation then allows determining the best subset of relay nodes for each source-destination pair.

The paper is organized as follows. The system model used in the analysis is described in Section II. The optimum operating bandwidth for the AF strategy is characterized in Section III and the optimum AF power allocation is derived in Section IV. In Section V, we address the DF forwarding strategy. The obtained results motivate consideration of a hybrid strategy that combines AF and DF and is presented in Section VI. This work builds on the previously presented results [17]–[19].
III. AF: OPTIMUM BANDWIDTH ALLOCATION

We now consider the rates achievable with the AF strategy. Let \( p_m \) denote the power at node \( m \) and let \( \phi = [p_1 \cdots p_M]^T \) be the power allocation at all nodes. Vector \( \phi = [p_1 \cdots p_M]^T \) denotes the nodes’ transmit powers per dimension and \( p_m = p_m/2W^{(i)}_m \) for \( m = 1, \ldots, M \). The amplification gain \( b_m \) is chosen such that the transmit power at node \( m \) is \( p_m \) and is found from (5) to be

\[
    b_m = \frac{p_m}{\alpha_mF_0 + N_0/2}, \quad m = 1, \ldots, M. \tag{7}
\]

The achievable rates are given by the mutual information between the channel input \( X_0[t] \) and the channel output, either \( Y^{(1)}[n] \) for shared relay bandwidth or \( Y^{(2)}[n] \) for orthogonal relay signaling. From results in [21], cases \( i = 1 \) and \( i = 2 \) can both be expressed in the form

\[
    I_{AF}^{(i)}(\phi) = \frac{1}{2} \log \left[ 1 + \frac{P_0}{N_0/2} (\beta_i + \epsilon^{(i)}(\phi)) \right], \tag{8}
\]

where \( \epsilon^{(i)}(\phi) \) represents a gain obtained by employing the AF relays. For shared bandwidth, it follows from (2), (3) and (5) that

\[
    \epsilon^{(1)}(\phi) = \sum_{m=1}^M \frac{\alpha_m\beta_0 P_m}{\alpha_m F_0 + N_0/2}, \tag{9}
\]

For orthogonal channels, (4) and (5) imply

\[
    \epsilon^{(2)}(\phi) = \sum_{m=1}^M \frac{\alpha_m\beta_m P_m}{\alpha_m F_0 + \beta_m P_m + N_0/2}. \tag{10}
\]

The rates (8) are normalized by the number of dimensions utilized by a node rather than by the total number of dimensions in the channel. For \( \epsilon^{(1)}(\phi) = 0 \), (8) becomes the rate achieved in the single-user channel, by a direct source transmission at power \( P_0 \). The difference in the AF gains (9) and (10) comes from the coherent combining of the relay signals transmitted in a shared bandwidth, which is forfeited in the orthogonal channel system. The analysis presented in this section, however, applies to both cases and we therefore drop the \( (i) \) superscript. We next consider the total rate achieved by the AF strategy

\[
    r_{AF} = 2W I_{AF}(\phi) \text{ bits/s,} \tag{11}
\]

where \( I_{AF} \) is given by (8). As \( W \) becomes large, we observe that \( G(\phi/W) \) decreases to zero and therefore

\[
    \lim_{W \to \infty} r_{AF} = \lim_{W \to \infty} W \log \left( 1 + \frac{\beta_0 P_0}{N_0 W} \right) = \frac{\beta_0 P_0}{N_0 \ln 2} \text{ bits/s}, \tag{12}
\]

which is the rate achieved in the wideband regime by the source transmission. Therefore, there is no benefit from AF relays transmitting in the wideband regime. This behavior was previously observed in [13] in a parallel Gaussian network with two relays. Except for the somewhat trivial case in which the source is in a favorable position compared to all the relays, the rate \( r_{AF} \) generally decreases for large \( W \).

To characterize the optimum AF bandwidth, we formulate the AF power/bandwidth relay problem as

\[
    r^* = \max_{\phi^*} \frac{2W I_{AF}(\phi)}{\alpha_m F_0 + N_0/2}, \tag{13a}
\]

subject to \( 2W \sum_{m=0}^M P_m \leq p \) \( \phi \geq 0, \tag{13b} \)

\[
    0 \leq W \leq W_{\text{max}}, \tag{13c}
\]

We assume that \( W_{\text{max}} \) is sufficiently large to allow the network to operate in the wideband regime. Let \( (\phi^*, W^*) \) denote the optimum power and bandwidth allocation that achieves \( r^* \) in (13). We first observe that, to achieve nonzero rate in (13), it has to hold that \( W^* > 0 \) and \( P_m > 0 \). Furthermore, constraint (13a) is always binding. Depending on the values of the channel gains, a solution to (13) may be a direct source transmission, that is, \( P_m^* = 0 \) for \( m = 1, \ldots, M \), \( W^* = W_{\text{max}} \) and \( P_m^* \) given by (13a). Otherwise, there will be a set of \( K \), \( 0 < K \leq M \), relays employing the AF strategy. In this case, the rate \( r_{AF} \) will be decreasing with \( W \) for large \( W \), implying that \( W^* < W_{\text{max}} \). Given \( \phi^* \), it will be convenient to relabel the nodes such that relays \( m \in \{1, \ldots, K\} \) are the active transmitters with powers \( P_m^* > 0 \) while \( P_m^* = 0 \), for \( m \in \{\{K+1, \ldots, M\} \). Since \( W^* > 0 \), the solution to (13) is never on the boundary (13c).

The Kuhn-Tucker conditions imply

\[
    \frac{\partial I_{AF}(\phi^*)}{\partial P_k} = \frac{I_{AF}(\phi^*)}{\sum_{m=0}^K P_m^*}, \quad k = 0, \ldots, K. \tag{14}
\]

From equations given by (14), we observe that the optimum power allocation \( P_m^* \) is independent of \( r \) and \( W^* \). We present the solution for the optimum relay powers in the next section. The optimum bandwidth can be determined such that the solution lies on the feasibility region boundary (13a)

\[
    W^* = \frac{p}{2 \sum_{m=0}^K P_m^*}, \tag{15}
\]

From (13), (14), and (15)

\[
    r^* = 2W^* I_{AF}(\phi^*) = \frac{I_{AF}(\phi^*)}{\sum_{m=0}^K P_m^*} p = \mu p. \tag{16}
\]

We thus proved the following:

**Theorem 1:** The AF relay problem (13) has an optimum solution in which the optimum bandwidth \( W^* \), the maximum rate \( r^* \) and the total power \( p \) have a linear relationship.

We can view \( \mu \) as a “rate reward,” or power efficiency; increasing the total available power in (13) by \( \Delta p \), increases the maximum achievable rate \( r^* \) by \( \mu \Delta p \).

IV. AF: OPTIMUM RELAY POWER ALLOCATION

We next consider a subproblem of (13) that determines the optimum relay powers per dimension, for any given source power \( P_0 \). We consider the shared bandwidth case first.

A. Shared Bandwidth

Given a source power \( P_0 \), we let

\[
    \gamma_m = \frac{\beta_m}{\alpha_m F_0 + N_0/2}, \tag{17}
\]
To maximize the rate (8) over the relay powers $\hat{\mathbf{P}} = [P_1 \cdots P_M]^T$, we maximize the AF gain (9)

$$\max_{\mathbf{P}} \frac{\left(\sum_{m=1}^{M} \sqrt{\alpha_m \gamma_m P_m}\right)^2}{\sum_{m=1}^{M} \gamma_m P_m + 1}$$

subject to $\sum_{m=1}^{M} P_m \leq P_R$.

$$\hat{\mathbf{P}} \geq 0$$

where $P_R = P/2W - P_0$ is the power allocated to the relays. To solve (18), we first argue that the solution is always on the boundary (18a). To see that, consider a feasible solution $\hat{\mathbf{P}}$ such that $\sum_{m=1}^{M} \hat{P}_m < P_R$. Then, there exist a constant $K > 1$ and a feasible solution $\mathbf{P}' = K\hat{\mathbf{P}}$ such that $\mathbf{P}'$ is on the boundary $K \sum_{m=1}^{M} \hat{P}_m = P_R$. Furthermore, it is easy to verify that $G(\mathbf{P}') > G(\hat{\mathbf{P}})$. We can, thus, let the constraint (18a) be satisfied with equality. The objective function (18) becomes

$$G(\mathbf{P}) = \frac{\left(\sum_{m=1}^{M} \sqrt{\alpha_m \gamma_m P_m}\right)^2}{\sum_{m=1}^{M} (\gamma_m + 1/P_R)P_m}.$$  

A solution to (18) can be found by introducing the vector $\mathbf{z} = [z_1 \cdots z_M]^T$ with

$$z_m = \sqrt{(\gamma_m + 1/P_R)P_m}, \quad m = 1, \ldots, M,$$

and a vector of coefficients $\mathbf{d} = [d_1 \cdots d_M]$ where

$$d_m = \sqrt{\frac{\alpha_m \gamma_m}{\gamma_m + 1/P_R}}, \quad m = 1, \ldots, M.$$  

Problem (18) can then be represented in a vector form

$$\max_{\mathbf{d}} (\mathbf{d}^T \mathbf{z})^2/\mathbf{z}^T \mathbf{z}.$$  

Applying the Schwarz inequality, the solution to (22) is $\mathbf{z}^* = k\mathbf{d}$ where the constant $k$ can be found from (18a) and (20). We get the optimum powers in the MRC form as

$$P_m^{\text{opt}} = \frac{P_R d_m}{\sum_{k=1}^{M} \delta_k},$$

where we define

$$\delta_m = \frac{\alpha_m \gamma_m}{(1+ \gamma_m P_R)^2}.$$  

The AF gain (9) becomes

$$G^{(1)}(P_0, P_R) = \sum_{m=1}^{M} \alpha_m \beta_m P_R + \frac{N_0}{2}.$$

The next Lemma follows by comparing AF gains (25) and (10).

Lemma 1: For any given source power $P_0$ and relay power $P_R$ with the relays employing AF signaling in shared bandwidth outperforms orthogonal signaling.

Given the relay powers (23) and AF gain (25), the shared-bandwidth AF power/bandwidth (13) with fixed bandwidth $W$ reduces to

$$\max_{P_0, P_R} I^{(1)}_{A^f}(P_0, P_R)$$

subject to $P_0 + P_R \leq \frac{P}{2W}$.

Lemma 2: There exists a unique optimum solution $(P_0^*(W), P_R^*(W))$ to (26).

The proof for the Lemma follows from the observation that the optimum is on the boundary where $P_R = P - P_0$ and that $I^{(1)}_{A^f}(P_0, P_R - P_0)$ is strictly concave in $P_0$.

Given $(P_0^*(W), P_R^*(W))$, the AF power/bandwidth (13) reduces to maximizing the rate with respect to bandwidth for $0 \leq W \leq W_{\text{max}}$

$$r^*(W) = \max_{\mathcal{W}} 2W I^{(1)}_{A^f}(P_0^*(W), P_R^*(W)).$$  

Numerical calculation of $r^*(W)$ is straightforward. The relay powers (23) are shown in Figs. 1–4 for a scenario of $M = 2500$ relays positioned on a $100 \times 100$ square grid. The source and the destination are positioned on the two opposite sides of the grid. The propagation exponent $\eta = 2$ was chosen.

For large source power $P_0$, relay powers are shown in Fig. 1. In this case, the received SNR at the relays is high and the network multiple-access side from the relays to the destination limits the performance. The relays that have a better channel to the destination are employed.

Fig. 2 shows the opposite case of a small power $P_0$ and a high power $P_R$. We observe a reversed relay power allocation compared to the previous case, as the network tries to improve the broadcast side performance by choosing the relays with high received SNR. Fig. 3 shows the powers for larger values of $P_0$ and $P_R$. Finally, Fig. 4 shows the relay powers when the network operates in a low SNR-regime due to small $P_0$ and $P_R$. The clustering behaviors shown in Figs. 1–4 are preserved as the number of relays in the network changes.
Fig. 2. Relay powers for $P_0 = 0.01$, $P_R = 10^4$. Relays employ AF in shared bandwidth. Relays that are closer to the source are employed.

Fig. 3. Relay powers for $P_0 = P_R = 10^4$ and $N_0/2 = 1$. Relays employ AF in shared bandwidth.

Fig. 4. Relay powers for $P_0 = P_R = 0.01$ and $N_0/2 = 1$. Relays employ AF in shared bandwidth.

Fig. 5. Achieved rate and optimum bandwidth as a function of the network power budget $p$ [Watts]. Relays employ AF in shared bandwidth.

B. Orthogonal Channels

We next identify the best subset of AF relays and their powers for the case of orthogonal signaling. Given a source power $P_0$, we let

$$\gamma_m = \sqrt{\frac{\alpha_m \beta_m}{\alpha_m P_0 + N_0/2}}.$$  

(28)

Again, to maximize the rate (8) over the relay powers $\hat{P}$, we maximize the AF gain (10)

$$\max_{\hat{P}} \sum_{m=1}^{M} \frac{\alpha_m \gamma_m^2 P_m}{\alpha_m + \gamma_m^2 P_m}$$

subject to

$$\sum_{m=1}^{M} P_m \leq P_R$$

(29a)

$$\hat{P} \geq 0.$$  

(29b)

From the Kuhn–Tucker conditions, the solution to (29) is in the water-filling form

$$P_m^* = \frac{\gamma_m}{\gamma_m} \left[ \frac{1}{\sqrt{\eta}} - \frac{1}{\gamma_m} \right]^+, \quad m = 1, \ldots, M$$  

(30)

where $\eta$ is the Lagrange multiplier and is found such that (29a) is satisfied with equality. Once again, the best choice of relays varies with the transmit source power. We observe that the AF relay network, depending on whether it operates in shared or orthogonal channels, will require two different relay power allocations as given by (23) or (30).

C. Single-Relay Channel

A different AF paradigm can be used in a single-relay channel (or in a relay network with multiple relays that cannot hear each others’ transmissions.) Under the assumption that a relay can transmit and receive simultaneously, we can allow the source and the relay to transmit at the same time in the shared bandwidth. As observed in [5], this strategy turns the relay channel into a unit-memory intersymbol interference channel, as the signal at the destination becomes

$$Y[n] = \sqrt{\beta_0}X[n] + \sqrt{\alpha_1\beta_1}X[n-1] + W[n],$$  

(31)

where $X[n]$ is the input and $W[n]$ is the noise.
and we say that source as a zero-mean Gaussian vector. We repeat the performance again when considering the channel (31) becomes the point-to-point channel with no benefit from the relay. Thus, this AF strategy can again benefit from the bandwidth optimization. We illustrate that fact on the network example of [5] where a source, relay and destination are positioned on a line. The source-relay distance is denoted as $d$. For $P_0 = P_1 = 10$, we repeat the performance comparison given in [5], in Fig. 6. Fig. 7 shows the comparison for $P_0 = P_1 = 0.01$. Note that, when employing DF strategy, a relay also transmits and receives simultaneously. We observe that the relative performance between DF and AF changes as the different power per dimension is used. Thus, in Fig. 8, we compare the two strategies while allowing each of them to operate in its optimum bandwidth and thus optimum power per dimension, for the given power (Watts) at the nodes.

V. DF

A multihopping strategy [5] in which data sent by the source is successively decoded by the relays and finally by the destination was shown to achieve the rates ([5, Thm. 1])

$$R_{DF} = \max_{\pi} \min_{0 \leq t \leq M} I(X^{\pi(0)}; Y^{\pi(t+1)} | X^{\pi(t+1:M)})$$  \hspace{1cm} (37)

where $\pi$ is a permutation on the set of nodes such that $\pi(0) = 0$, $\pi(M + 1) = M + 1$ and $\pi(i : j) = (\pi(i), \pi(j))$, and $X_{i:j}$ denotes the channel inputs $X_{i:j} = [X_i, X_{i+1}, \ldots, X_j]$. For a fixed covariance matrix $R = E[XX^T]$, it follows from the conditional maximum entropy theorem ([22, Lemma 1]) that all the terms in (37) are maximized by choosing $X$ as a zero-mean Gaussian vector.

The two-hop DF is a more constrained case of multihopping and it imposes a constraint on the correlation between the inputs. The rate (37) then reduces to the minimum of the broadcast rate achieved in the first hop and the MAC rate achieved in the second hop from the relays to the destination. With $A(R_0)$ denoting the subset of relays executing the DF strategy and $\alpha = \min_{j \in A(R_0)} \{\alpha_j\}$, the first hop broadcast rate is

$$R_{BC} = \frac{1}{2} \log \left( 1 + \frac{\alpha P_0}{N_0/2} \right)$$  \hspace{1cm} (38)

and the second hop MAC rate is

$$R_{MAC}^{(1)} = \frac{1}{2} \log \left( 1 + \frac{1}{N_0/2} \left( \beta_0 P_0 + \left( \sum_{j \in A(R_0)} \sqrt{\beta_j P_j} \right)^2 \right) \right) \hspace{1cm} (39)$$

The channel capacity from the source to any node in $A(R_0)$ is thus higher than the code rate $R_{BC}$ and we say that source makes a node in $A(R_0)$ reliable. In general, the source power is split in two parts: the first for transmission to the relays, and the second for helping the relays forward a message to the destination [2]. However, in (38) and (39), the source power is used for
the first goal exclusively. The reason is that the MAC rate (39), increases with $M$ and thus will become higher than the rate (38) for sufficiently large $M$.

In the case of the orthogonal signaling, the MAC rate is given by

$$R_{\text{MAC}}^{(2)} = \frac{1}{2} \log \left( 1 + \frac{P_0/\beta_0}{N_0/2} \right) + \sum_{j \in A(P_0)} \frac{1}{2} \log \left( 1 + \frac{P_j/\beta_j}{N_0/2} \right).$$

(40)

As in the AF case, the difference in the two bandwidth allocations is that the signaling in the common bandwidth allows for the coherent combining of the relay signals at the destination. The achievable rate (37) reduces to

$$I_{\text{DF}}^{(i)} = \min \left\{ R_{\text{BC}}, R_{\text{MAC}}^{(i)} \right\}, \quad i = 1, 2.$$  

(41)

The achievable rate (41) is bounded by the worst source-relay link and by the MAC part of the relay network that, for any $A(P_0)$, is a Gaussian vector channel [21] with relays acting as a multiple-antenna transmitter. For the given powers, the maximum rate in bits/s or equivalently, the minimum energy cost per information bit in both the point-to-point and Gaussian vector channel is achieved in the limit of large $W$ [15]. Thus, the power-efficiency of DF strategy is maximized in the wideband regime. This behavior was also analyzed in [16].

A. DF Orthogonal Signaling: Optimum Power Allocation

In this case, the MAC rate is given by $R_{\text{MAC}}^{(2)}$. For the given power $P_0$ and the rate $r$ at the source, relay $m$ will be able to execute the DF strategy only if the rate $r$ can be communicated reliably from the source to relay $m$ with power $P_0$. Thus, it has to hold that

$$W \log \left( 1 + \frac{\alpha_m P_0}{N_0/2} \right) \geq r.$$  

(42)

When constraint (42) is met for node $m$, we say that the source makes node $m$ reliable. To optimize the transmit powers, we have to find the best subset of nodes to be made reliable so that they can DF the message. We use binary variables $x_i$ to indicate which relays $i$ will be in the active set $A(P_0)$ and formulate the maximization of $2W I_{\text{DF}}^{(2)}$ in the following way:

$$\begin{alignat*}{2}
\max \quad & r = W \log \left( 1 + \frac{P_0/\beta_0}{N_0/2} \right) \\
& + \sum_{i=1}^M x_i W \log \left( 1 + \frac{P_i/\beta_i}{N_0/2} \right)
\end{alignat*}$$

subject to

$$\begin{alignat*}{2}
W \log \left( 1 + \frac{P_0 x_i/\alpha_i}{N_0/2} \right) & > x_i r \\
\sum_{i=0}^M x_i p_i & \leq p \\
x_i & \in \{0, 1\} \\
p_i & \geq 0.
\end{alignat*}$$

(43)

Specifically, (43) sets $R_{\text{MAC}}^{(2)} = r$ while (43a) requires that rate $r$ be achievable at each active relay. In the limit of large $W$, (43) simplifies to the orthogonal wideband single relay problem

$$\begin{alignat*}{2}
\max \quad & r = p_0/\beta_0 + \sum_{i=1}^M x_i p_i/\beta_i \\
\text{subject to} \quad & p_0 x_i > x_i r \\
& \sum_{i=0}^M p_i \leq p \\
& x_i \in \{0, 1\} \\
& p_i \geq 0.
\end{alignat*}$$

(44)

From (44), we observe that in terms of the set $A(P_0) = \{x_i = 1\}$ of active relays

$$r = p_0/\beta_0 + \sum_{i \in A(P_0)} x_i p_i = p_0/\beta_0 + \left( \sum_{i \in A(P_0)} p_i \right) \max_{i \in A(P_0)} \beta_i.$$

(45)

Moreover, this upper bound is achievable by assigning the relay power budget $\sum_{i \in A(P_0)} p_i$ to a single relay $k$ with $\beta_k = \max_{i \in A(P_0)} \beta_i$. This observation yields the following claim.

**Theorem 2:** The orthogonal wideband DF relay problem (44) admits an optimal solution in which no more than one relay node transmits.

Thus, the intuition of Theorem 2 is that the relays provide a set of parallel channels to the destination and under wideband operation, transmitted power per dimension is severely restricted. Thus, waterfilling this power over the relay channels results in transmission only on the best channel to the destination.

By Theorem 2, it is sufficient to consider only policies that employ a single relay $k$. In this case, $x_k = 1$, and $x_i = 0$ for $i \neq k$. Equation (44) becomes the wideband single relay problem

$$\begin{alignat*}{2}
\max \quad & r_k = p_0/\beta_0 + p_k/\beta_k \\
\text{subject to} \quad & p_0 x_k \geq r_k \\
& p_0 + p_k \leq p \\
& p_0, p_k \geq 0
\end{alignat*}$$

(46)

In (46), one can show that relay $k$ is used with power $p_k > 0$ only if $x_k > 0$ and $\beta_k > \beta_0$. In this case, the transmit powers are

$$\begin{alignat*}{2}
p_k^* &= \frac{\beta_k}{\alpha_k + \beta_k - \beta_0} p_k \\
p_0^* &= \frac{\alpha_k - \beta_0}{\alpha_k + \beta_k - \beta_0} p_k
\end{alignat*}$$

(47)

The achieved rate normalized by the noise variance

$$r_k^* = \frac{\alpha_k \beta_k}{\alpha_k + \beta_k - \beta_0} p_k.$$  

(48)
We emphasize that this is the optimal power assignment for using node $k$ as long as node $k$ is a useful relay, in the sense that $k$ belong to the set of useful relays

$$U = \{k | \alpha_k > \beta_0, \beta_k > \beta_0\}. \quad (49)$$

Finally, among all useful relays $k$, we choose that one which maximizes the rate $r_k^\ast$. We summarize our observations in the following theorem.

**Theorem 3:** If the set $U$ of useful relays is nonempty, the optimal solution to the orthogonal wideband DF relay problem (44) is for the source to employ relay

$$k^\ast = \arg\min_{k \in U} \left[ \frac{1}{\alpha_k} + \frac{1}{\beta_k} - \frac{\beta_0}{\alpha_k \beta_0} \right] \quad (50)$$

with power assignment given by (47); otherwise, if $U$ is empty, then direct transmission from the source to the destination is optimal.

**Remark 1:** Employing a single relay out of the set of available relays, as in Thm. 2, was more recently shown to be superior also in fading, in terms of the outage probability [23]. From a more practical aspect, this approach was considered in [24].

### B. DF Coherent Combining: Optimum Power Allocation

When the DF relays share the bandwidth, the maximum rate problem can be formulated as

$$\begin{align*}
\max \quad & r = W \log \left( 1 + \frac{p_0 \beta_0}{WN_0} + \frac{\sum_{i=1}^{M} \sqrt{x_i \beta_i p_i}}{WN_0} \right)^2 \\
\text{subject to} \quad & W \log \left( 1 + \frac{p_0 \beta_0}{WN_0} \right) > x_i r \\
& \sum_{i=0}^{M} p_i \leq p \\
& x_i \in \{0, 1\} \\
& p_i \geq 0,
\end{align*} \quad (51)$$

In the limit of large $W$, this problem simplifies to the wideband DF relay problem

$$\begin{align*}
\max \quad & r = p_0 \beta_0 + \left( \sum_{i=1}^{M} \sqrt{x_i \beta_i p_i} \right)^2 \\
\text{subject to} \quad & p_0 \beta_0 > x_i r \\
& \sum_{i=0}^{M} p_i \leq p \\
& x_i \in \{0, 1\} \\
& p_i \geq 0,
\end{align*} \quad (52)$$

Any choice of source power $p_0$ determines a reliable set of relays $A(p_0)$, for which (42) is satisfied. With total power $p_r = p - p_0$ allocated to relays, (52) simplifies to determining the optimum powers $\hat{p}_i$ of relays within the set $A(p_0)$

$$\begin{align*}
\max_{\hat{p}_i} \quad & \left( \sum_{i \in A(p_0)} \sqrt{\beta_i \hat{p}_i} \right)^2 \\
\text{subject to} \quad & \sum_{i=1}^{M} \hat{p}_i \leq p_r \\
& \hat{p}_i \geq 0,
\end{align*} \quad (53)$$

Since relays $i \not\in A(p_0)$ do not contribute to the rate objective (53), the optimization problem sets those relay powers to zero. For the reliable relays $i \in A(p_0)$, it is straightforward to show that the solution is in the MRC form

$$\hat{p}_i^\ast = \frac{\beta_i p_r}{\sum_{k \in A(p_0)} \beta_k}, \quad i \in A(p_0). \quad (54)$$

Thus, unlike the orthogonal DF case, each reliable relay is employed in order to contribute to the coherent combining gain. The corresponding achievable rate (52) is

$$r(p_0) = \beta_0 p_0 + p_r \sum_{i \in A(p_0)} \beta_i. \quad (55)$$
Without loss of generality, we can assume that the relay nodes are labeled such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_M$. Thus, if $p_0 \alpha_k \geq r$, then relay nodes 1 through $k$ will be able to decode. From (55), the achievable rate will become

$$r(p_0) = \beta_0 p_0 + p_k c_k$$  \hspace{1cm} (56)$$

where

$$c_k = \sum_{i=1}^{k} \beta_i.$$  \hspace{1cm} (57)$$

From (52) and (56), the wideband relay problem reduces to

$$\max_{p_k} r_k = \beta_0 p_0 + p_k c_k \hspace{1cm} (58)$$

subject to

$$p_0 \alpha_k > r_k, \hspace{1cm} (58a)$$

$$p_0 + p_k \leq p, \hspace{1cm} (58b)$$

$$p_0, p_k \geq 0. \hspace{1cm} (58c)$$

We observe that (58) is identical to (46) with $c_k$ replacing $\beta_k$. In this case, however, node $k$ will not be the only transmitting relay, but rather the transmitting relay with the $k$th largest link gain to the source.

Using the same reasoning as in the case of (46), we conclude that a set of relays $\{1, \ldots, k\}$ is employed if $\alpha_k > \beta_0$ and $c_k > c_0$ for a given $p_0$. From (58), we obtain the optimum powers

$$p_0^* = \alpha_k + c_k - \beta_0 p_0 \hspace{1cm} p_k^* = \alpha_k - \beta_0.$$  \hspace{1cm} (59)$$

The set of useful relays in this case is given by

$$U_c = \{k|\alpha_k > \beta_0, c_k > c_0\}. \hspace{1cm} (60)$$

We choose $k^*$ such that

$$k^* = \arg \min_{k \in U_c} \left[ \frac{1}{\alpha_k} + \frac{1}{c_k} - \frac{\beta_0}{\alpha_k c_k} \right]. \hspace{1cm} (61)$$

C. Numerical Comparison

We compare the performance of the AF and DF strategies in a network of size $10 \times 10$. In Figs. 9, 10, and 11 a different number of relays ($10$, $100$, and $1000$, respectively) are randomly positioned in the network. We consider the case when the relays share the bandwidth. For the AF strategy, we choose $P_0 = P_{R}$. We observe that, for a small number of relays, the DF strategy performs better than AF. As the number of relays increases, the AF strategy catches up with DF and ultimately outperforms it. The limitation of DF comes from the constraint that relays have to decode the message and, thus, DF cannot fully utilize the relays. This effect becomes more prominent as the number of relays gets large.

VI. HYBRID STRATEGY

In the DF strategy, the signaling rate is limited by a channel from the source to the active relay with the smallest channel gain. This limitation is overcome by allowing AF at the relays and thus relaxing the decoding constraint. In this case, however, a part of relay power is wasted to amplify the receiver noise. In this section, we propose a hybrid strategy, in which a relay amplifies and forwards a signal only if it cannot reliably decode the source message. Otherwise, a relay employs DF. This scheme is expected to perform better than pure AF as a subset of nodes will be forwarding a clear signal without unnecessary amplification of noise.

A. Optimum Power Allocation

For a fixed signaling bandwidth $W$, source power $P_0$ and total relay power $P_R$, we can use results from Section IV-A to derive the optimum relay power allocation for the hybrid strategy. For a given source power $P_0$ the set of reliable relays $A(R_0)$ is given by (42). For the achievable rate (10), it is then straightforward to determine the gain as in (9). In particular, the problem can be expressed in a form equivalent to (18) as

$$\max_{P} \left\{ \frac{\sum_{m \notin A(R_0)} \sqrt{\gamma_m P_m}}{\sum_{m \in A(R_0)} \gamma_m P_m} + \sum_{m \in A(R_0)} \sqrt{\gamma_m P_m} \right\}$$

subject to

$$\sum_{m=1}^{M} P_m \leq P_R, \hspace{1cm} (62a)$$

$$\hat{P} \geq 0. \hspace{1cm} (62b)$$

where $P_0 = P_{R}/2$ is the power allocated to the relays and $\gamma_m$ is given by (17). We have

$$\gamma_m = \frac{\beta_m}{\gamma_m \alpha_m + N_0/2}, \hspace{1cm} (63)$$

Note that when $A(R_0)$ is an empty set, (62) reduces to (18). In fact, we let

$$\gamma_m = \begin{cases} \frac{\beta_m}{\alpha_m P_0 + N_0/2} & \text{if } m \notin A(R_0) \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (64)$$

$$\nu_m = \begin{cases} \frac{\beta_m}{\alpha_m P_0} & \text{if } m \notin A(R_0) \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (65)$$

so that the gain (62) is reduced to

$$\frac{\left(\sum_{m=1}^{M} \sqrt{\gamma_m \gamma_m + \nu_m \gamma_m} \right)^2}{\sum_{m=1}^{M} \gamma_m P_m + 1}. \hspace{1cm} (66)$$
Repeating steps (19)–(25), we obtain the solution
\[ P^* = \frac{I_R R_h}{\sum_{k=1}^{M} \delta_k H} \]
where we defined
\[ \delta_k H = \frac{\alpha_m (\sqrt{m} + \sqrt{P_m})^2}{(1 + \gamma_m R_h)^2}. \]
When \( A(P_0) \) is an empty set, solution reduces to the AF solution (23); similarly when all \( A(P_0) \) contains all the nodes, the solution reduces to the DF solution (55).

VII. CONCLUSION
The results presented in this paper indicate show that the choice of a coding strategy goes beyond determining a coding scheme at a node; it also determines the operating bandwidth as well as the best distribution of the relay power. While we consider a single source-destination pair, our results have implications for networks with multiple source-destination pairs. Our view is that the relay network between each source-destination pair is a resource that we aim to use efficiently. Such a view motivates a total power constraint as the network budget. The optimum power allocation then allows determining the best subset of relay nodes for each source-destination pair. The obtained results assume full knowledge of channel gains at the nodes in order to bring insights into operating networks with relays. One future path would be to examine the impact of reduced channel state information available at the nodes.

In a wireless network, messages are typically expected to travel further than just two hops and the two-hop protocol approach should not be viewed as an obstacle to multipathing protocols. In fact, it is expected that a routing protocol will still operate on the network layer. The cooperative relay strategies will be run on the lower MAC layer, allowing for faster network adaptation to changes due to fading or high mobility. In that sense, routing and relaying will work together to increase the network performance.

REFERENCES

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