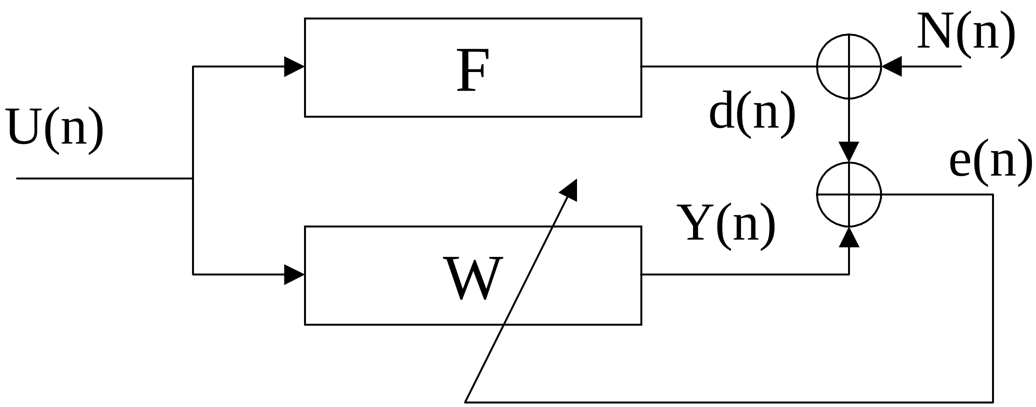
Optimal Filtering



Focus on linear, discrete time filters using mean square error as the statistical criterion for optimization

The above model covers equalizer, echo canceller, and linear predictors

Optimal Filter Theory

- Wiener Filtering, spectral factorization:

Estimation error

$$e(n) = d(n) - \sum w_k^* u(n-k)$$

Cost function:

$$\varepsilon = E\{e(n)e^*(n)\} = E\{|d(n)|^2\} - \sum_{k=0}^{M-1} w_k^* \rho\ (-k) - \sum_{k=0}^{M-1} w_k \rho^*(-k) + \sum_{k=0}^{M-1}\sum_{i=0}^{M-1} w_k^* w_i r(i-k)$$

$$r(i-k) = E\{u(n-k)u^*(n-i)\}$$

$$\rho(i) = E\{u(n+k)d^*(n)\}$$

Minimizing the cost function:

$$\frac{\partial \varepsilon}{\partial w_k} = \rho(-k) - \sum_{i=0}^{M-1} w_{oi} r(i-k) = 0 \quad k = 0,1,\cdots,M-1$$

Optimum coefficients:

$$\sum_{i=0}^{M-1} w_{oi} r(i-k) = \rho(-k) \quad k = 0, \cdots, M-1$$

In matrix form:

$$RW_o = P$$

$$R = \begin{pmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r^*(1) & r(0) & & r(M-2) \\ \vdots & & \vdots & \\ r^*(M-1) & r^*(M-2) & & r(0) \end{pmatrix}$$

$$P = [\rho(0), \rho(-1), \cdots, \rho(1-M)]^T$$

$$W = [w_{o0}, w_{o1}, \cdots, w_{oM-1}]$$

$$\varepsilon = \sigma_d^2 - W^H P - P^H W + W^H R W$$

The optimum solution is

$$W_o = R^{-1} P$$

$$\varepsilon_{\min} = E\{|d(n)|^2\} - P^H W_o$$

$$\varepsilon = \varepsilon_{\min} + (w - w_o)^H R (w - w_o)$$

Canonical form

$$R = Q \Lambda Q^H$$

$$\upsilon = Q^H (w - w_o)$$

$$\varepsilon = \varepsilon_{\min} + \sum_{k=1}^{M} \lambda_k \upsilon_k \upsilon_k^* = \varepsilon_{\min} + \sum_{k=1}^{M} \lambda_k |\upsilon_k|^2$$

The new coefficient vector contains the principle axes of error-performance surface.

- Steepest descent Algorithm: Iterative solution to Wiener equation

$$w(n+1) = w(n) + \mu[-\nabla(\varepsilon)]$$

$$w(n+1) = w(n) + \mu[P - RW(n)]$$

- Gradient operator

$$\nabla\varepsilon(W) = [\frac{\partial\varepsilon(W)}{\partial w_0} \cdots \frac{\partial\varepsilon(W)}{\partial w_{M-1}}]$$

$$= -2P + 2RW(n)$$

- The correction is the expectation of inner product of input vector and estimation error

$$\nabla(\varepsilon(W)) = 2E[u(n-k)e^*(n)]$$

Stability analysis:

$$c(n) = w(n) - w_0(n)$$

Then:

$$c(n+1) = (I - \mu R)c(n)$$

$$R = Q\Lambda Q^H$$

$$c(n+1) = (I - \mu Q\Lambda Q^H)c(n)$$

Define: $\upsilon(n) = Q^H c(n)$

Then

$$\upsilon(n+1) = (I - \mu\Lambda)\upsilon(n)$$

For the kth natural mode:

$$\upsilon_k(n) = (1 - \mu\lambda_k)^n \upsilon_k(0)$$

Stability requirement:

$$-1 < 1 - \mu\lambda_k < 1 \Rightarrow 0 < \mu < \frac{2}{\lambda_{\max}}$$