A Unified Approach for Calculating Error Rates of Linearly Modulated Signals over Generalized Fading Channels

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***Abstract*— We present a unified analytical framework to determine the exact average symbol-error rate (SER) of linearly modulated signals over generalized fading channels. The results are applicable to systems employing coherent demodulation with maximal-ratio combining multichannel reception. The analyses assume independent fading paths, which are not necessarily identically distributed. In all cases, the proposed approach leads to an expression of the average SER involving a single finite-range integral, which can be easily computed numerically. In addition, as special cases, SER expressions for single-channel reception are obtained. These expressions reduce to well-known solutions, give alternative (often simpler) expressions for previous results, or provide new formulas that are either closed-form expressions or simple to compute numerically.**

***Index Terms*— Error-rate calculation, linear modulations, maximal-ratio combining diversity, multipath fading channels.**

I. Introduction

Wireless communication systems are subject to severe multipath fading that can seriously degrade their performance. Thus, fading compensation is typically required to mitigate the effect of multipath. Diversity combining [1], which combines multiple replicas of the received signal, is a classical and powerful technique to combat multipath impairment. Space diversity, achieved by using multiple antennas at the receiver, is the most common form of diversity. Diversity can also be implemented for wide-band systems over frequency-selective fading channels using RAKE reception [2], [3]. The main idea of RAKE reception is to combine resolvable multipath components in order to increase the received signal-to-noise ratio (SNR).

There are many papers dealing with the performance of linear coherent modulation over fading channels [4]–[20]. When multichannel reception is considered with some special exceptions [14]–[17] most of the models for these systems typically assume either Rayleigh paths or independently, identically distributed (i.i.d.) Nakagami or Rician paths. These idealizations are not always realistic since the average fading power [21], [22] and the severity of fading [23]–[25] may vary from one path to the other when, for example, multipath diversity is employed. In this paper, we consider a sufficiently general multilink channel model in which the paths are not necessarily identically distributed nor even distributed according to the same family of distributions. We call these channels generalized fading channels, and we describe them in more detail in Section II-B. We derive expressions for the exact symbol-error rate (SER) of linearly modulated signals over such channels. The results are applicable to systems that employ coherent demodulation and maximal-ratio combining (MRC) [1]. The proposed approach takes advantage of alternative integral representations [26]–[28] of the probability of error of these signals over additive white Gaussian noise (AWGN) channels (i.e., the conditional SER), along with some known Laplace transforms and/or Gauss–Hermite quadrature integrals, to derive the SER expressions. These expressions involve a single finite-range integral whose integrand contains only elementary functions and that can, therefore, be easily evaluated.

alternative representation of the Gaussian (tail probability) Q-function has been used by Femenias and Furió [29]–[32], Hall and Wilson [33], and Simon and Divsalar [34] to solve several other problems involving the performance of coded communication systems over Rayleigh and Nakagami-m fading channels. In addition, Tellambura *et al.* [19] published a recent paper in which they used these representations to analyze the performance of M-ary phase-shift keying (M-PSK) with MRC diversity reception. This work, which was done independently, has some of the same features as our approach, however, it focuses on a smaller set of modulation techniques (M-PSK only) and a smaller set of channel models/conditions (i.i.d. diversity paths with multipath and shadow fading statistics). In this paper, we use the alternative representations to unify and add to the results cited above by providing new generic expressions for the average SER performance of various coherent communication systems with MRC diversity reception over generalized fading channels. Since the number of different modulation and fading combinations discussed herein is quite large, numerical results for the error rates of these combinations and dependence on

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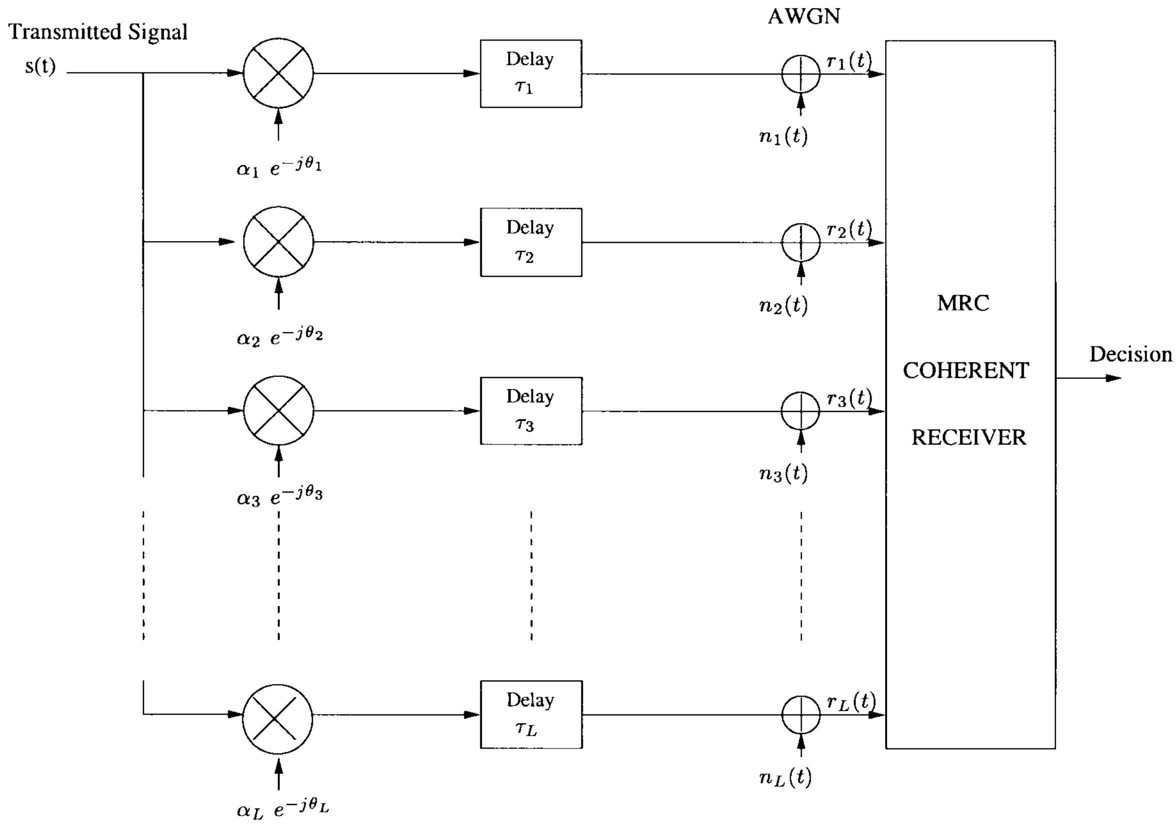

Fig. 1. Multilink channel model.

the various fading parameters are omitted here. Some of the referenced papers cover such numerical illustrations. A more comprehensive treatment that includes numerical results can be found in [35].

The remainder of this paper is organized as follows. In the next section, we describe the transmitted signals, introduce the generalized multipath channel model, and present the receiver under consideration. We derive the average bit-error rate (BER) of binary signals over generalized fading channels in Section III. The average SER of M-PSK, M-ary amplitude modulation (M-AM), and square M-ary quadrature amplitude modulation (M-QAM) signals over generalized fading channels is described in Sections IV–VI, respectively, using an approach similar to that of Section III. Last, we give a summary of our results and offer some concluding remarks in Section VII.

II. System and Channel Models

A. Transmitted Signals

With any memoryless linear modulation technique, the complex signal transmitted over the channel may be represented as

$$s(t) = \sum_{i=-\infty}^{\infty} S^{(i)} e^{-j2\pi f_c t} P_{T_s}(t - iT_s) \tag{1}$$

where the function $P_{T_s}(\cdot)$ is a pulse shaping waveform of duration T_s seconds, f_c is the carrier frequency, and $\{S^{(i)}\}_{i=-\infty}^{\infty}$ represents the sequence of symbols that results from mapping successive k-bit blocks into one of $M = 2^k$ possible waveforms. Each complex symbol $S^{(i)}$ takes on values whose energy is denoted by E_m $(m = 1, 2, \cdots, M)$ and the average energy per k-bit symbol (averaged over the set of the M waveforms' energies) is denoted by E_s and is related to the average energy per bit E_b by $E_s = \log_2(M) E_b$.

B. Channel Model

We consider a multilink channel where the transmitted signal is received over L independent slowly-varying flat fading channels, as shown in Fig. 1. In Fig. 1, l is the channel index, and $\{\alpha_l\}_{l=1}^{L}$, $\{\theta_l\}_{l=1}^{L}$, and $\{\tau_l\}_{l=1}^{L}$ are the random channel amplitudes, phases, and delays, respectively. We assume that the sets $\{\alpha_l\}_{l=1}^{L}$, $\{\theta_l\}_{l=1}^{L}$, and $\{\tau_l\}_{l=1}^{L}$ are mutually independent. The first channel is assumed to be the reference channel with delay $\tau_1 = 0$ and, without loss of generality, we assume that $\tau_1 < \tau_2 < \cdots < \tau_L$. Because of the slow-fading assumption, we assume that the $\{\alpha_l\}_{l=1}^{L}$, $\{\theta_l\}_{l=1}^{L}$, and $\{\tau_l\}_{l=1}^{L}$ are all constant over a symbol interval.

The fading amplitudes $\{\alpha_l\}_{l=1}^{L}$ are assumed to be statistically independent random variables (RV's) whose mean square value $\overline{\alpha_l^2}$ is denoted by Ω_l and whose probability density function (pdf) is any of the family of distributions described below. The multilink channel model used in our analyses is sufficiently general to include the case where the different channels are not necessarily identically distributed nor even distributed according to the same family of distributions. We call this type of multilink channel a generalized multilink fading channel.

After passing through the fading channel, each replica of the signal is perturbed by complex AWGN with a one-sided power spectral density, which is denoted by $2N_l$ (W/Hz). The AWGN is assumed to be statistically independent from channel to channel and independent of the fading amplitudes $\{\alpha_l\}_{l=1}^{L}$. Hence, the instantaneous SNR per symbol of the lth channel is given by $\gamma_l = (\alpha_l^2 E_s)/N_l$, where $E_s(J)$ is the energy per symbol.

Now, we briefly present the different fading pdf's considered in our analyses. Note that a more detailed treatment of this particular topic will be presented in [35, Ch. 2]. We are including many fading models in our analysis for two reasons. First, we want to show that our unified approach is applicable to general channels with arbitrary fading distributions on each diversity branch. Second, we want to compute the moment-generating functions (MGF's) for the most common distributions encountered in practice, so that other researchers and engineers could use our results to easily compute average SER for M-PSK, M-AM, or M-QAM signals.

1) Multipath Fading: Multipath fading is due to the constructive and destructive combination of randomly delayed, reflected, scattered, and diffracted signal components. Depending on the nature of the radio propagation environment, there are different models describing the statistical behavior of the multipath fading envelope.

a) Rayleigh: The Rayleigh distribution is frequently used to model multipath fading with no direct line-of-sight (LOS) path. In this case, the lth channel fading amplitude α_l is distributed according to

$$p_{\alpha_l}(\alpha_l; \Omega_l) = \frac{2\alpha_l}{\Omega_l} \exp\left(-\frac{\alpha_l^2}{\Omega_l}\right), \qquad \alpha_l \geq 0 \tag{2}$$

and, hence, the instantaneous SNR per symbol of the lth channel γ_l is distributed according to an exponential distribution given by

$$p_{\gamma_l}(\gamma_l; \bar{\gamma}_l) = \frac{1}{\bar{\gamma}_l} \exp\left(-\frac{\gamma_l}{\bar{\gamma}_l}\right), \qquad \gamma_l \geq 0 \tag{3}$$

where $\bar{\gamma}_l = (\Omega_l E_s)/N_l$ denotes the average SNR per symbol of the lth channel.

b) Nakagami-q (Hoyt): The Nakagami-q distribution, also referred to as the Hoyt distribution [36], is given in [37, eq. (52)] by

$$p_{\alpha_l}(\alpha_l; \Omega_l, q_l) = \frac{(1+q_l^2)\alpha_l}{q_l \Omega_l} \exp\left(-\frac{(1+q_l^2)^2 \alpha_l^2}{4q_l^2 \Omega_l}\right) \cdot I_0\left(\frac{(1-q_l^4)\alpha_l^2}{4q_l^2 \Omega_l}\right), \qquad \alpha_l \geq 0 \tag{4}$$

where $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind, and q_l is the Nakagami-q fading parameter that ranges from 0 to 1. Using a change of variables, it can be shown that the SNR per symbol of the lth channel γ_l is distributed according to

$$p_{\gamma_l}(\gamma_l; \bar{\gamma}_l, q_l) = \frac{(1+q_l^2)}{2q_l \bar{\gamma}_l} \exp\left(-\frac{(1+q_l^2)^2 \gamma_l}{4q_l^2 \bar{\gamma}_l}\right) \cdot I_0\left(\frac{(1-q_l^4)\gamma_l}{4q_l^2 \bar{\gamma}_l}\right), \qquad \gamma_l \geq 0. \tag{5}$$

The Nakagami-q distribution spans the range from one-sided Gaussian fading $(q_l = 0)$ to Rayleigh fading $(q_l = 1)$.

c) Nakagami-n (Rice): The Nakagami-n distribution is also known as the Rice distribution [38]. It is often used to model propagation paths consisting of one strong direct LOS component and many random weaker components. Here, the lth channel fading amplitude follows the distribution [37, eq. (50)]

$$p_{\alpha_l}(\alpha_l; \Omega_l, n_l) = \frac{2(1+n_l^2)e^{-n_l^2}\alpha_l}{\Omega_l} \exp\left(-\frac{(1+n_l^2)\alpha_l^2}{\Omega_l}\right) \cdot I_0\left(2n_l \alpha_l \sqrt{\frac{1+n_l^2}{\Omega_l}}\right), \qquad \alpha_l \geq 0 \tag{6}$$

where n_l is the Nakagami-n fading parameter that ranges from 0 to ∞ and is related to the Rician K_l factor by $K_l = n_l^2$. Here, the SNR per symbol of the lth channel γ_l is distributed according to a noncentral chi-square distribution given by

$$p_{\gamma_l}(\gamma_l; \bar{\gamma}_l, n_l) = \frac{(1+n_l^2)e^{-n_l^2}}{\bar{\gamma}_l} \exp\left(-\frac{(1+n_l^2)\gamma_l}{\bar{\gamma}_l}\right) \cdot I_0\left(2n_l \sqrt{\frac{(1+n_l^2)\gamma_l}{\bar{\gamma}_l}}\right), \qquad \gamma_l \geq 0. \tag{7}$$

The Nakagami-n distribution spans the range from Rayleigh fading $(n_l = 0)$ to no fading (constant amplitude) $(n_l = \infty)$.

d) Nakagami-m: The Nakagami-m pdf is in essence a central chi-square distribution given by [37, eq. (11)]

$$p_{\alpha_l}(\alpha_l; \Omega_l, m_l) = \frac{2m_l^{m_l}\alpha_l^{2m_l-1}}{\Omega_l^{m_l}\Gamma(m_l)} \exp\left(-\frac{m_l \alpha_l^2}{\Omega_l}\right), \qquad \alpha_l \geq 0 \tag{8}$$

where $\Gamma(\cdot)$ is the gamma function, and m_l is the Nakagami-m fading parameter that ranges from $1/2$ to ∞. In this case, the SNR per symbol γ_l of the lth channel is distributed according to a gamma distribution given by

$$p_{\gamma_l}(\gamma_l; \bar{\gamma}_l, m_l) = \frac{m_l^{m_l}\gamma_l^{m_l-1}}{\bar{\gamma}_l^{m_l}\Gamma(m_l)} \exp\left(-\frac{m_l \gamma_l}{\bar{\gamma}_l}\right), \qquad \gamma_l \geq 0. \tag{9}$$

The Nakagami-m distribution spans via the m parameter the widest range of fading among all the multipath distributions considered in this paper. For instance, it includes the one-sided Gaussian distribution $(m_l = 1/2)$ and the Rayleigh distribution $(m_l = 1)$ as special cases. In the limit as $m_l \rightarrow +\infty$, the Nakagami-m fading channel converges to a nonfading AWGN channel.

2) Log-Normal Shadowing: In terrestrial and satellite land-mobile systems, the link quality is also affected by slow variation of the mean signal level due to the shadowing from terrain, buildings, and trees. Communication system performance will depend only on shadowing if the radio receiver is able to average out the fast multipath fading or if an efficient "micro"-diversity system is used to eliminate the effects of multipath. Based on empirical measurements, there is a general consensus that shadowing can be modeled by a log-normal distribution for various outdoor and indoor environments [39, Sec. 2.4], in which case the lth path SNR per symbol γ_l has a pdf given by the standard log-normal expression

$$p_{\gamma_l}(\gamma_l;\mu_l,\sigma_l)=\frac{10}{\ln 10\sqrt{2\pi}\sigma_l\gamma_l}\exp\left[-\frac{(10\log_{10}\gamma_l-\mu_l)^2}{2\sigma_l^2}\right] \tag{10}$$

where μ_l (decibels) and σ_l (decibels) are the mean and the standard deviation of $10\log_{10}\gamma_l$, respectively. We assume flat multipath fading in this section. In the next section, we consider composite multipath/shadowing channels.

3) Composite Multipath/Shadowing: A composite multipath/shadowed fading environment consists of multipath fading superimposed on log-normal shadowing. In this environment, the receiver does not average out the envelope fading due to multipath, but rather reacts to the instantaneous composite multipath/shadowed signal [39, Sec. 2.4.2]. This is typically the scenario in congested downtown areas with slow moving pedestrians and vehicles [40]–[42]. This type of composite fading is also observed in land-mobile satellite systems subject to vegetative and/or urban shadowing [43]–[47]. There are two approaches and various combinations suggested in the literature for obtaining the composite distribution. Here, as an example, we present the composite gamma/log-normal pdf introduced by Ho and Stüber [42]. This pdf arises in Nakagami-m shadowed environments and is obtained by averaging the gamma-distributed signal power (or equivalently the SNR per symbol) (9) over the conditional density of the log-normally distributed mean signal power (or equivalently, the average SNR per symbol) (10), giving the following pdf for the lth channel:

$$p_{\gamma_l}(\gamma_l;\mu_l,m_l,\sigma_l)=\int_0^\infty\frac{m_l^{m_l}\gamma_l^{m_l-1}}{w^{m_l}\Gamma(m_l)}\exp\left[-\frac{m_l\gamma_l}{w}\right]\frac{10}{\ln 10\sqrt{2\pi}\sigma_l w}\cdot\exp\left[-\frac{(10\log_{10}w-\mu_l)^2}{2\sigma_l^2}\right]dw. \tag{11}$$

For the special case where the multipath is Rayleigh distributed $(m_l=1)$, (11) reduces to a composite exponential/log-normal pdf which was initially proposed by Hansen and Meno [41].

4) Combined (Time-Shared) Shadowed/Unshadowed: From their land-mobile satellite channel characterization experiments, Lutz *et al.* [46] and Barts and Stutzman [48] found that the overall fading process for land-mobile satellite systems is a convex combination of unshadowed multipath fading and a composite multipath/shadowed fading. Here, as an example, we present in more detail the Lutz *et al.* model [46]. When no shadowing is present, the fading follows a Rice (Nakagami-n) pdf. On the other hand when shadowing is present, it is assumed that no direct LOS path exists and the received signal power (or equivalently, SNR per bit) is assumed to follow an exponential/log-normal (Hansen–Meno) pdf [41]. The combination is characterized by the shadowing time-share factor that is denoted by A, $0\le A\le 1$. The resulting combined pdf is given by

$$p_{\gamma_l}\left(\gamma_l;A_l,\bar{\gamma}_l^u,K_l;\mu_l^s,\sigma_l\right)=(1-A_l)p_{\gamma_l}\left(\gamma_l;\bar{\gamma}_l^u,n_l=\sqrt{K_l}\right)+A_lp_{\gamma_l}\left(\gamma_l;1,\mu_l^s,\sigma_l\right) \tag{12}$$

where $\bar{\gamma}_l^u$ is the average SNR per symbol during the unshadowed fraction of time, and μ_l^s is the average of $10\log_{10}\gamma_l$ during the shadowed fraction of time. The overall average SNR per symbol $\bar{\gamma}_l$ is then given by

$$\bar{\gamma}_l=(1-A_l)\bar{\gamma}_l^u+A_l10^{((\mu_l^s/10)+(\ln 10)\sigma_l^2/200)}. \tag{13}$$

C. MRC Receiver

We assume an L branch (finger) MRC receiver. This form of diversity combining is optimal, since it results in the maximum-likelihood receiver [39, p. 244]. For equally-likely transmitted symbols, the total conditional SNR per symbol γ_t at the output of the MRC combiner is given by [39, p. 246, eq. (5.98)]

$$\gamma_t=\sum_{l=1}^L\gamma_l. \tag{14}$$

III. Average BER of Binary Signals

A. Product Form Representation of the Conditional BER

The user's conditional BER $P_b(E\mid\{\gamma_l\}_{l=1}^L)$ is given by

$$P_b(E\mid\{\gamma_l\}_{l=1}^L)=Q(\sqrt{2g\gamma_t}) \tag{15}$$

where $g=1$ for coherent binary phase-shift keying (BPSK) [49, eq. (4.55)], $g=1/2$ for coherent orthogonal binary frequency-shift keying (BFSK) [49, eq. (4.59)], $g=0.715$ for coherent BFSK with minimum correlation [49, eq. (4.63)], and $Q(\cdot)$ is the Gaussian Q-function traditionally defined by

$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\,dt. \tag{16}$$

Although (15) appears to be a very simple expression, it is often inconvenient when further analyses are required. In particular, our goal is to evaluate the performance of the system in terms of users' average BER, and for this purpose, the conditional BER (15) has to be statistically averaged over the random parameters $\{\gamma_l\}_{l=1}^L$. This requires the integration of the Gaussian Q-function over these parameters, which is difficult since the argument of the function is in the *lower limit of the integral*. The classical approach to bypass this problem is to first find the pdf of γ_t and then average (15) over that pdf. In some cases of i.i.d. channels, the pdf of γ_t can be found, which then often leads to simple closed-form expressions for the average BER. However, it is more difficult to find a simple expression for the pdf of γ_t when the channels have the same distribution (e.g., Nakagami-n [Rice]) but with different parameters (e.g., different average fading powers and/or different fading parameters). The most difficult case occurs when the pdf's of the different channels come from different families of distributions, and in this case, finding the pdf of γ_t appears intractable.

The key concept in our approach is to rely on an alternative representation of the Gaussian Q-function. This representation allows us to obtain an elegant analytical expression for the average BER of the generalized multilink channel model, which heretofore resisted a simple solution. The alternative representation was proposed by Craig who showed that the Gaussian Q-function could be represented in the following definite integral form [28, eq. (9)]:

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\phi}\right) d\phi, \qquad x \geq 0 \tag{17}$$

which can also be implied by the earlier work of Weinstein [26] and by Pawula *et al.* [27]. A simple derivation of this alternative representation of the Gaussian Q-function is given in Appendix A-1. This representation has the advantage of finite integration limits that are *independent of the argument* x, and it also has an integrand that is *Gaussian in the argument* x. Using the alternative representation of the Gaussian Q-function (17) in (15), the conditional BER (15) may be rewritten in a more desirable *product form* given by

$$\begin{aligned} P_b\left(E \mid \{\gamma_l\}_{l=1}^L\right) &= \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{g\gamma_t}{\sin^2\phi}\right) d\phi \\ &= \frac{1}{\pi}\int_0^{\pi/2} \prod_{l=1}^{L} \exp\left(-\frac{g\gamma_l}{\sin^2\phi}\right) d\phi. \end{aligned} \tag{18}$$

This form of the conditional BER is more desirable, since we can first independently average over the individual statistical distributions of the γ_l's and then perform the integral over ϕ, as described in more detail below.

B. Average BER With Single-Channel Reception $(L = 1)$

Since the fading is assumed to be independent of the AWGN, the unconditional BER $P_b(E)$ is obtained by averaging the single-channel conditional BER $P_b(\gamma)$, given by (18) for $L = 1$, over the underlying fading RV giving

$$P_b(E) = \int_0^{\infty} P_b(E \mid \gamma) p_\gamma(\gamma; \bar{\gamma}, i)\, d\gamma \tag{19}$$

where i is the fading parameter(s) associated with the distribution $p_\gamma(\gamma; \bar{\gamma}, i)$ and is, hence, denoted by r,[1] q, n, m, σ, $m\sigma$, and $AK\sigma$ for the Rayleigh, Nakagami-q (Hoyt), Nakagami-n (Rice), Nakagami-m, log-normal shadowing, composite multipath/shadowing, and combined (time-shared) shadowed/unshadowed pdf's, respectively. Substituting (18) for $L = 1$ into (19), then interchanging the order of integration, yields

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2} \mathcal{M}_i\left(-\frac{g}{\sin^2\phi}; \bar{\gamma}\right) d\phi \tag{20}$$

where

$$\mathcal{M}_i(s; \bar{\gamma}) \triangleq \int_0^{\infty} p_\gamma(\gamma; \bar{\gamma}, i) e^{s\gamma}\, d\gamma \tag{21}$$

is the MGF of the SNR per symbol and is in the form of a Laplace transform. The form of the average BER in (20) is interesting in that the MGF's can either be obtained in closed-form with the help of classical Laplace transforms or can alternatively be efficiently computed by using Gauss–Hermite quadrature integration [50, p. 890, eq. (25.4.46)] for all previously mentioned fading channel models. We now evaluate these integrals for each of the fading models described in Section II-B. In Section II-C, we will use these integrals to obtain the average BER of binary signals with multichannel reception.

1) Multipath Fading

a) Rayleigh: Substituting (3) into (21), then using the Laplace transform [51, p. 1178, eq. (1)]

$$\int_0^{\infty} e^{-sx}\, dx = \frac{1}{s}, \qquad s > 0 \tag{22}$$

yields

$$\mathcal{M}_r\left(-\frac{g}{\sin^2\phi}; \bar{\gamma}\right) = \left(1 + \frac{g\bar{\gamma}}{\sin^2\phi}\right)^{-1}. \tag{23}$$

Substituting (23) in (20), then using [51, p. 185, eq. (2.562.1)], one can proceed further to obtain the well-known closed-form expression for the average BER over Rayleigh fading [3, eqs. (7.3.7), (7.3.8)]

$$P_b(E) = \frac{1}{2}\left(1 - \sqrt{\frac{g\bar{\gamma}}{1 + g\bar{\gamma}}}\right). \tag{24}$$

b) Nakagami-q (Hoyt): Substituting (5) into (21), then using the Laplace transform [51, p. 1182, eq. (109)]

$$\int_0^{\infty} I_0(ux) e^{-sx}\, dx = (s^2 - u^2)^{-1/2}, \qquad s > |u| \geq 0 \tag{25}$$

yields

$$\mathcal{M}_q\left(-\frac{g}{\sin^2\phi}; \bar{\gamma}\right) = \left(1 + \frac{2g\bar{\gamma}}{\sin^2\phi} + \frac{4q^2g^2\bar{\gamma}^2}{(1+q^2)^2\sin^4\phi}\right)^{-1/2}. \tag{26}$$

c) Nakagami-n (Rice): Substituting (7) into (21), then using the Laplace transform [50, p. 1026, eq. (29.3.81)]

$$\int_0^{\infty} I_0(u\sqrt{x}) e^{-sx}\, dx = \frac{e^{u^2/(4s)}}{s}, \qquad s > 0 \tag{27}$$

yields

$$\begin{aligned} \mathcal{M}_n\left(-\frac{g}{\sin^2\phi}; \bar{\gamma}\right) = & \frac{(1+n^2)\sin^2\phi}{(1+n^2)\sin^2\phi + g\bar{\gamma}} \\ & \cdot \exp\left(-\frac{n^2 g\bar{\gamma}}{(1+n^2)\sin^2\phi + g\bar{\gamma}}\right). \end{aligned} \tag{28}$$

[1] Note that for Rayleigh fading, the pdf has no dependency on r, and the symbol r is just used to identify the Rayleigh case.

d) Nakagami-m: Substituting (9) into (21), then using the Laplace transform [51, p. 1178, eq. (3)]

$$\int_0^\infty x^\nu e^{-sx}\,dx = \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \qquad s>0, \quad \nu>-1 \tag{29}$$

yields

$$\mathcal{M}_m\left(-\frac{g}{\sin^2\phi};\bar{\gamma}\right) = \left(1+\frac{g\bar{\gamma}}{m\sin^2\phi}\right)^{-m}. \tag{30}$$

As a side result, we show in Appendix B that by substituting (30) in (20) and then using an equivalence with a known result, we obtain a closed-from expression for trigonometric integrals, which do not exist in classical tables of integrals such as [50], [51]. These integrals can be used to simplify calculations involving for example the performance of BPSK and M-PSK with selection diversity over correlated Nakagami-m fading channels [52].

2) Log-Normal Shadowing: If the channel statistics follow a log-normal distribution, it is straightforward to show that $\mathcal{M}_\sigma(s;\mu)$ can be accurately approximated by Gauss–Hermite integration yielding

$$\mathcal{M}_\sigma\left(-\frac{g}{\sin^2\phi};\mu\right) \simeq \frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p} H_{x_n}\exp\left(-\frac{g10^{(\sqrt{2}\sigma x_n+\mu)/10}}{\sin^2\phi}\right) \tag{31}$$

where N_p is the order of the Hermite polynomial $H_{N_p}(\cdot)$. Setting N_p to 20 is typically sufficient for excellent accuracy. In (31), x_n are the zeros of the N_p-order Hermite polynomial, and H_{x_n} are the weight factors of the N_p-order Hermite polynomial and are given by

$$H_{x_n} = \frac{2^{N_p-1}N_p!\sqrt{\pi}}{N_p^2 H_{N_p-1}^2(x_n)}. \tag{32}$$

Both the zeros and the weights factors of the Hermite polynomial are tabulated in [50, p. 924, Table (25.10)] for various polynomial orders N_p.

3) Composite Multipath/Shadowing: If the channel statistics follow a gamma/log-normal distribution, it is straightforward to show that the MGF $\mathcal{M}_{m\sigma}(s;\mu)$ can be accurately evaluated by using (29) followed by a Gauss–Hermite integration yielding

$$\mathcal{M}_{m\sigma}\left(-\frac{g}{\sin^2\phi};\mu\right) \simeq \frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p} H_{x_n}\left(1+\frac{g10^{(\sqrt{2}\sigma x_n+\mu)/10}}{m\sin^2\phi}\right)^{-m}. \tag{33}$$

4) Combined (Time-Shared) Shadowed/Unshadowed: If the channel statistics follow a combined Lutz *et al.* distribution, it is straightforward to show that the MGF $\mathcal{M}_{AK\sigma}(s;\bar{\gamma}^u,\mu^s)$ can be broken into two terms, one that can be evaluated in closed-form and the other that can be accurately approximated by Gauss–Hermite integration, yielding

$$\mathcal{M}_{AK\sigma}\left(-\frac{g}{\sin^2\phi};\bar{\gamma}^u,\mu^s\right) \simeq (1-A)\mathcal{M}_n\left(-\frac{g}{\sin^2\phi};\bar{\gamma}^u\right)+A\mathcal{M}_{m\sigma}\left(-\frac{g}{\sin^2\phi};\mu^s\right) \tag{34}$$

with $n=\sqrt{K}$ in $\mathcal{M}_n(s;\bar{\gamma}^u)$ and $m=1$ in $\mathcal{M}_{m\sigma}(s;\mu^s)$.

C. Average BER With Multichannel Reception $(L>1)$

To obtain the unconditional BER $P_b(E)$, when multichannel reception is used, we must average the multichannel conditional BER $P_b(E \mid \{\gamma_l\}_{l=1}^L)$ over the joint pdf of the instantaneous SNR sequence $\{\gamma_l\}_{l=1}^L$, namely $p_{\gamma_1,\gamma_2,\cdots,\gamma_L}(\gamma_1,\gamma_2,\cdots,\gamma_L)$. Since the RV's $\{\gamma_l\}_{l=1}^L$ are assumed to be statistically independent, then $p_{\gamma_1,\gamma_2,\cdots,\gamma_L}(\gamma_1,\gamma_2,\cdots,\gamma_L) = \prod_{l=1}^L p_{\gamma_l}(\gamma_l;\bar{\gamma}_l,i_l)$, and the averaging procedure results in

$$P_b(E) = \underbrace{\int_0^\infty\int_0^\infty\cdots\int_0^\infty}_{L-\text{fold}} P_b(\{\gamma_l\}_{l=1}^L) \cdot\prod_{l=1}^L p_{\gamma_l}(\gamma_l;\bar{\gamma}_l,i_l)\,d\gamma_1\,d\gamma_2\cdots d\gamma_L \tag{35}$$

where i_l represents the fading parameter(s) associated with the lth channel. Note that if the traditional integral representation of the Gaussian Q-function (16) were to be used in the $P_b(E \mid \{\gamma_l\}_{l=1}^L)$ term, (35) would result in an $L+1$-fold integral with infinite limits (one of these integrals comes from the classical definition of the Gaussian Q-function (16) in $P_b(E \mid \{\gamma_l\}_{l=1}^L)$), and a closed-form solution or an adequately efficient numerical integration method would not be available. Using the alternative product form representation of the conditional BER (18) in (35) yields

$$P_b(E) = \underbrace{\int_0^\infty\int_0^\infty\cdots\int_0^\infty}_{L-\text{fold}} \frac{1}{\pi}\int_0^{\pi/2}\prod_{l=1}^L \exp\left(-\frac{g\gamma_l}{\sin^2\phi}\right) \cdot p_{\gamma_l}(\gamma_l;\bar{\gamma}_l,i_l)\,d\phi\,d\gamma_1\,d\gamma_2\cdots d\gamma_L. \tag{36}$$

The integrand in (36) is absolutely integrable, and hence, the order of integration can be interchanged. Thus, grouping terms of index l, we obtain

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2}\prod_{l=1}^L \mathcal{M}_{i_l}\left(-\frac{g}{\sin^2\phi};\bar{\gamma}_l\right)d\phi \tag{37}$$

where $\mathcal{M}_{i_l}(s;\bar{\gamma}_l)$ is the MGF of the SNR per symbol associated with path l and is given above for the various channel models.[2] If the fading is identically distributed with the same fading parameter i and the same average SNR per bit $\bar{\gamma}$ for all L channels, then (37) reduces to

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2}\left(\mathcal{M}_i\left(-\frac{g}{\sin^2\phi};\bar{\gamma}\right)\right)^L d\phi. \tag{38}$$

Hence, in all cases this approach reduces the $L+1$-fold integral with infinite limits of (35) (accounting for the infinite range integral coming from the traditional representation of the Gaussian Q-function) to a single finite-range integral (37) whose integrand contains only elementary functions, such as

[2]Recall that the approach presented in this paper applies to independent diversity channels. Although some of the "features" of this approach also apply to correlated diversity channels [53], independent fading paths for microdiversity systems (antenna arrays) is unlikely in the presence of large-scale fading effects, such as shadowing. In this case, the analysis presented in this paper would be limited to macro-diversity systems.

exponentials and trigonometrics, and can, therefore, be easily evaluated numerically.

It is interesting to mention at this point that the same final result (37) can be obtained, without using the alternative representation of the Gaussian Q-function, but by starting with [8, eq. (17)]. Indeed it has been pointed out to the authors by Mazo [54] that [8, eq. (17)] which is expressed in terms of the characteristic function of γ_t (using our notations) can be rewritten in terms of the MGF of γ_t by changing the integration contour. The details of the procedure are described in an internal AT&T Bell Laboratories memorandum that was never submitted for publication [55]. Following that procedure and using the fact that the MGF of the sum of independent RV's is the product of the MGF of the individual RV's [56, Sec. 7.4], [8, eq. (17)] can be rewritten as (using again our notations)

$$P_b(E) = \frac{1}{2\pi}\int_1^{\infty} \frac{\prod_{l=1}^{L} \mathcal{M}_i(-gy;\bar{\gamma}_l)}{y\sqrt{y-1}}\,dy \qquad (39)$$

which can be changed to the same single finite-range integral (37) by adopting the change of variables $y = \frac{1}{\sin^2\phi}$ [54].

IV. Average SER of M-PSK Signals

A. Product Form Representation of the Conditional SER

The conditional SER for M-PSK $P_s(E \mid \{\gamma_l\}_{l=1}^L)$ does not exist in closed-form. However, it can be shown that it is given exactly by the desirable integral expression [27, eq. (71)], [28, eq. (5)], [49, eq. (3.119)]

$$\begin{aligned} P_s(E \mid \{\gamma_l\}_{l=1}^L) &= \frac{1}{\pi}\int_0^{((M-1)\pi)/M} \exp\left(-\frac{g_{\text{PSK}}\gamma_t}{\sin^2\phi}\right) d\phi \\ &= \frac{1}{\pi}\int_0^{((M-1)\pi)/M} \prod_{l=1}^{L} \exp\left(-\frac{g_{\text{PSK}}\gamma_l}{\sin^2\phi}\right) d\phi \qquad (40) \end{aligned}$$

where $g_{\text{PSK}} = \sin^2(\pi/M)$.

B. Average SER of M-PSK

Following the same steps as in (35)–(37), it can be easily shown that the average SER of M-PSK $P_s(E)$ over generalized fading channels is given by

$$P_s(E) = \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \prod_{l=1}^{L} \mathcal{M}_{i_l}\left(-\frac{g_{\text{PSK}}}{\sin^2\phi};\bar{\gamma}_l\right) d\phi. \qquad (41)$$

Our result (41) generalizes the M-PSK average SER results of [4, eq. (22)] and [10, eq. (21)] for L i.i.d. Rayleigh paths. It also gives an alternative approach for the performance evaluation of coherent M-PSK over frequency-selective channels characterized by a Rician dominant path with Rayleigh secondary paths [15], [17]. Furthermore, by setting L to 1, the result (41) can be used to evaluate the average SER performance of M-PSK with single-channel reception. This leads, for example, to the following results:

- *Rayleigh:* Substituting (23) in (41) (with $L = 1$), then using [51, p. 185, eq. (2.562.1)] yields a closed-form expression [6, eq. (9)], [7, eq. (7)] for the SER of M-PSK over a Rayleigh channel, which agrees with the results obtained using various other methods [4, eq. (22)], [11, eq. (36)].
- *Nakagami-n (Rice):* Substituting (28) in (41) leads to an expression for the SER of M-PSK over a Nakagami-n (Rice) channel, which is easily shown to agree with [11, eq. (35)].
- *Nakagami-m:* Substituting (30) in (41) (with $L = 1$) gives the SER of M-PSK over a Nakagami-m channel as

$$P_s(E) = \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \left(1 + \frac{\bar{\gamma}\sin^2\left(\frac{\pi}{M}\right)}{m\sin^2\phi}\right)^{-m} d\phi. \qquad (42)$$

Note that (42) yields the same numerical values as [5, eq. (17)] and [14, eq. (9)], and it is much easier to compute for any arbitrary value of m.

V. Average SER of M-AM Signals

A. Product Form Representation of the Conditional SER

The conditional SER for M-AM $P_s(E \mid \{\gamma_l\}_{l=1}^L)$ with signal points symmetrically located about the origin is given by [49, p. 631]

$$P_s(E \mid \{\gamma_l\}_{l=1}^L) = \frac{2(M-1)}{M} Q(\sqrt{2g_{\text{am}}\gamma_t}) \qquad (43)$$

where $g_{\text{am}} = 3/(M^2-1)$. Using the alternative representation of the Gaussian Q-function (17) in (43), we obtain the conditional SER in the desired product form as

$$\begin{aligned} P_s(E \mid \{\gamma_l\}_{l=1}^L) &= \frac{2(M-1)}{M\pi}\int_0^{\pi/2} \exp\left(-\frac{g_{\text{am}}\gamma_t}{\sin^2\phi}\right) d\phi \\ &= \frac{2(M-1)}{M\pi}\int_0^{\pi/2} \prod_{l=1}^{L} \exp\left(-\frac{g_{\text{am}}\gamma_l}{\sin^2\phi}\right) d\phi. \end{aligned} \qquad (44)$$

B. Average SER of M-AM

Following the same steps as in (35)–(37), it is straightforward to show that the average SER of M-AM over generalized fading channels is given by

$$P_s(E) = \frac{2(M-1)}{M\pi}\int_0^{\pi/2} \prod_{l=1}^{L} \mathcal{M}_{i_l}\left(-\frac{g_{\text{am}}}{\sin^2\phi};\bar{\gamma}_l\right) d\phi. \qquad (45)$$

VI. Average SER of Square M-QAM Signals

A. Product Form Representation of the Conditional SER

Consider square M-QAM signals whose constellation size is given by $M = 2^k$ with k even. The conditional SER for square M-QAM is given by [49, eq. (10.32)]

$$\begin{aligned} P_s(E \mid \{\gamma_l\}_{l=1}^L) = {} & 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{2g_{\text{QAM}}\gamma_t}\right) \\ & \cdot - 4\left(1 - \frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{2g_{\text{QAM}}\gamma_t}\right) \qquad (46) \end{aligned}$$

where $g_{\mathrm{QAM}} = 3/(2(M-1))$. Simon and Divsalar [34] generalized the alternative representation of the Gaussian Q-function to the two-dimensional case and showed in particular that [34, eq. (80)]

$$Q^2(x) = \frac{1}{\pi}\int_0^{\pi/4} \exp\left(-\frac{x^2}{2\sin^2\phi}\right) d\phi, \qquad x \geq 0. \quad (47)$$

A simple proof of this result is given in Appendix A-2. Using the alternative representation of the Gaussian Q-function (17), as well as the new representation of the square of the Gaussian Q-function (47), the conditional SER (46) may be rewritten in the more desirable product form given by

$$\begin{aligned} P_s\left(E\left|\{\gamma_l\}_{l=1}^L\right.\right) &= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\exp\left(-\frac{g_{\mathrm{QAM}}\gamma_t}{\sin^2\phi}\right)d\phi \\ &\quad - \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\exp\left(-\frac{g_{\mathrm{QAM}}\gamma_t}{\sin^2\phi}\right)d\phi \\ &= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\prod_{l=1}^{L}\exp\left(-\frac{g_{\mathrm{QAM}}\gamma_l}{\sin^2\phi}\right)d\phi \\ &\quad - \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\prod_{l=1}^{L}\exp\left(-\frac{g_{\mathrm{QAM}}\gamma_l}{\sin^2\phi}\right)d\phi. \end{aligned}$$

B. Average SER of M-QAM

Following the same steps as in (35)–(37) yields the average SER of M-QAM over generalized fading channels as

$$\begin{aligned} P_s(E) &= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\prod_{l=1}^{L}\mathcal{M}_{i_l}\left(-\frac{g_{\mathrm{QAM}}}{\sin^2\phi};\bar{\gamma}_l\right)d\phi \\ &\quad - \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\prod_{l=1}^{L}\mathcal{M}_{i_l}\left(-\frac{g_{\mathrm{QAM}}}{\sin^2\phi};\bar{\gamma}_l\right)d\phi. \end{aligned} \quad (48)$$

Of particular interest is the average SER performance of M-QAM with single-channel reception, which can be obtained by setting L to 1 in (48). For example, substituting (23) in (48) (with $L=1$), then again using [51, p. 185, eq. (2.562.1)] yields a closed-form expression for the average SER of M-QAM over Rayleigh channels as

$$\begin{aligned} P_s(E) &= 2\left(1-\frac{1}{\sqrt{M}}\right)\left(1-\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}}\right)+\left(1-\frac{1}{\sqrt{M}}\right)^2 \\ &\quad \cdot\left[\frac{4}{\pi}\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}}\arctan\left(\sqrt{\frac{1+g_{\mathrm{QAM}}\bar{\gamma}}{g_{\mathrm{QAM}}\bar{\gamma}}}\right)-1\right]. \end{aligned} \quad (49)$$

Note that (49) matches the result obtained by [11, eq. (44)] for the particular case where $M=16$. Note also that (49) can in fact be obtained alternatively by averaging (46) over the Rayleigh pdf (2) and by using a standard known integral involving the function $\mathrm{erfc}^2(\cdot)$ [51, p. 941, eq. (8.258.2)]. In addition, using [35, eqs. (5A.4b) and (5A.21)] in (48), we obtain the performance of M-QAM over L i.i.d. Rayleigh fading channels as

$$\begin{aligned} P_s(E) &= 4\left(1-\frac{1}{\sqrt{M}}\right)\left(\frac{1-\mu_c}{2}\right)^L\sum_{l=0}^{L-1}\binom{L-1+l}{l} \\ &\quad \cdot\left(\frac{1+\mu_c}{2}\right)^L - 4\left(1-\frac{1}{\sqrt{M}}\right)^2 \\ &\quad \cdot\left(\frac{1}{4}-\frac{\mu_c}{\pi}\left(\left[\frac{\pi}{2}-\arctan\mu_c\right]\sum_{l=0}^{L-1}\frac{\binom{2l}{l}}{4(1+g_{\mathrm{QAM}}\bar{\gamma})^l}\right.\right. \\ &\quad - \sin(\arctan\mu_c)\sum_{l=1}^{L-1}\sum_{i=1}^{l}\frac{T_{il}}{(1+g_{\mathrm{QAM}}\bar{\gamma})^l} \\ &\quad \left.\left.\cdot\left[\cos(\arctan\mu_c)\right]^{2(l-i)+1}\right)\right) \end{aligned} \quad (50)$$

where

$$\mu_c = \sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}}{1+g_{\mathrm{QAM}}\bar{\gamma}}} \quad (51)$$

and

$$T_{il} = \frac{\binom{2l}{l}}{\binom{2(l-i)}{l-i}4^i(2(l-i)+1)}. \quad (52)$$

Note that (50) is equivalent to the expressions [18, eq. (15)] and [20, eq. (12)], which involves a sum of Gauss hypergeometric functions.[3] Furthermore, using a partial fraction expansion on the integrand of (48), we obtain with the help of [51, p. 185, eq. (2.562.1)] the average SER of M-QAM over L Rayleigh fading channels with distinct average fading powers and with MRC reception as

$$\begin{aligned} P_s(E) &= 2\left(1-\frac{1}{\sqrt{M}}\right)\sum_{l=1}^{L}\rho_l\left(1-\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}_l}{1+g_{\mathrm{QAM}}\bar{\gamma}_l}}\right) \\ &\quad + \left(1-\frac{1}{\sqrt{M}}\right)^2\left[\frac{4}{\pi}\sum_{l=1}^{L}\rho_l\sqrt{\frac{g_{\mathrm{QAM}}\bar{\gamma}_l}{1+g_{\mathrm{QAM}}\bar{\gamma}_l}}\right. \\ &\quad \left.\cdot\arctan\left(\sqrt{\frac{1+g_{\mathrm{QAM}}\bar{\gamma}_l}{g_{\mathrm{QAM}}\bar{\gamma}_l}}\right)-\sum_{l=1}^{L}\rho_l\right] \end{aligned}$$

where

$$\rho_l = \left(\prod_{\substack{k=1\\k\neq l}}^{L}\left(1-\frac{\bar{\gamma}_k}{\bar{\gamma}_l}\right)\right)^{-1} \quad (53)$$

which is equivalent to the expressions [18, eq. (10)] and [20, eq. (21)].

[3] Equation (12) in [20] gives the same numerical result as the one given by (50) if a minor typo is corrected in [20, eq. (18)] (the denominator should be $(2k+1)\sqrt{\pi}$ rather than $(2k-1)\sqrt{\pi}$).

VII. Conclusion

We have presented a unified analytical framework to determine the exact average SER of linearly modulated signals over generalized fading channels. The results are applicable to systems employing coherent demodulation with MRC multichannel reception. The multichannel model is sufficiently general to include paths that are not necessarily identically distributed nor even distributed according to the same family of distributions.

The unified framework is achieved by exploiting alternative integral representations of the conditional probability of error in which the conditional SNR is inside the integrand rather than in the limit of integration. This, combined with closed-form Laplace transforms and/or Gauss–Hermite quadrature integrations, leads to expressions of the average SER that involve a single finite-range integral whose integrand contains only elementary functions and can, therefore, be easily computed numerically. In addition, we presented as special cases average SER expressions for single-channel reception. These expressions reduce to well-known solutions, give alternative (often simpler) expressions for previous results, or provide new formulas which are either closed-form expressions or simple to evaluate numerically.

Appendix A
Derivations of the Alternate Representations of the Gaussian Q-Function and its Square

A byproduct of Craig's work on the probability of error for two-dimensional signal constellations [28] was the alternative representation of the Gaussian Q-function given in (17). An extension of this representation for the square of the Gaussian Q-function (47) was obtained by Simon and Divsalar [34]. In this appendix, we present another simple method of proving the alternative representations of the Gaussian Q-function and its square.

1. Proof of (17)

The proposed proof is an extension of the classical method to evaluate the Laplace–Gauss integral [51, eq. (3.321.3)]

$$J(a) \triangleq \int_0^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}, \qquad a>0. \tag{54}$$

Let us consider the double integral

$$\int_0^{\infty}\int_x^{\infty} e^{-\frac{u^2+v^2}{2}}\,du\,dv, \qquad x\ge 0. \tag{55}$$

Because of separability, (55) can be rewritten as

$$\underbrace{\int_0^{\infty} e^{-u^2/2}\,du}_{J(1/\sqrt{2})}\underbrace{\int_x^{\infty} e^{-v^2/2}\,dv}_{\sqrt{2\pi}Q(x)} = \pi Q(x) \tag{56}$$

where we see that each integral in the LHS of (56) is a well-defined function. Further, transformation to polar coordinates $u=r\cos\phi$ and $v=r\sin\phi$ $(du\,dv = r\,dr\,d\phi)$ may be carried out in (55) giving

$$\begin{aligned}\int_0^{\infty}\int_x^{\infty} e^{-(u^2+v^2)/2}\,du\,dv &= \int_0^{\pi/2}\int_{x/\sin\phi}^{\infty} e^{-r^2/2}r\,dr\,d\phi\\ &= \int_0^{\pi/2}\exp\left(-\frac{x^2}{2\sin^2\phi}\right)d\phi.\end{aligned} \tag{57}$$

Equating the RHS of (56) and (57), we obtain an alternative proof of the desired result (17). Note that another purely algebraic proof of the result (17), which can be implied from the work of Pawula *et al.* [27], is given in detail in [35, Appendix 4-A].

2. Proof of (47)

The proof presented in Appendix A-1 can be easily extended to arrive at the alternative representation of $Q^2(\cdot)$ given in (47). Let us now consider the following double integral

$$\int_x^{\infty}\int_x^{\infty} e^{-\frac{u^2+v^2}{2}}\,du\,dv, \qquad x\ge 0. \tag{58}$$

Again, because of separability, (58) can be rewritten as

$$\underbrace{\int_x^{\infty} e^{-u^2/2}\,du}_{\sqrt{2\pi}Q(x)}\underbrace{\int_x^{\infty} e^{-v^2/2}\,dv}_{\sqrt{2\pi}Q(x)} = 2\pi Q^2(x) \tag{59}$$

where each integral in the LHS of (59) is the Gaussian Q-function multiplied by $\sqrt{2\pi}$. The transformation to polar coordinates $u=r\cos\phi$ and $v=r\sin\phi$ $(du\,dv=r\,dr\,d\phi)$ is carried out in (58), and by symmetry, the rectangular region of integration is divided into two equal triangular parts giving

$$\begin{aligned}\int_x^{\infty}\int_x^{\infty} e^{-\frac{u^2+v^2}{2}}\,du\,dv &= 2\int_0^{\pi/4}\int_{x/\sin\phi}^{\infty} e^{-r^2/2}r\,dr\,d\phi\\ &= 2\int_0^{\pi/4}\exp\left(-\frac{x^2}{2\sin^2\phi}\right)d\phi.\end{aligned} \tag{60}$$

Equating (59) and (60), we obtain an alternative proof of the Simon–Divsalar result (47).

Appendix B
Closed-Form Expressions for $\int_0^{\pi/2}\left(\frac{\sin^2\phi}{\sin^2\phi+c}\right)^m d\phi$

The alternative representation of the Gaussian Q-function can also be used to find closed-form expressions for integrals not tabulated in classical table of integrals such as [50], [51]. As an example, we evaluate in this appendix the integral $I_m(c)$ defined by

$$I_m(c) \triangleq \int_0^{\pi/2}\left(\frac{\sin^2\phi}{\sin^2\phi+c}\right)^m d\phi. \tag{61}$$

To do so, consider first the integral $J_m(a,b)$ defined by

$$J_m(a,b) \triangleq \frac{a^m}{\Gamma(m)}\int_0^{+\infty} e^{-at}t^{m-1}Q(\sqrt{bt})\,dt, \qquad m\ge 0. \tag{62}$$

This integral (62) has a known closed-form expression. When m is a positive real number, the integral $J_m(a,b)$ is given by [57, eq. (A8)]

$$J_m(a,b) \triangleq J_m(c) = \frac{\sqrt{c/\pi}}{2(1+c)^{m+1/2}} \frac{\Gamma(m+1/2)}{\Gamma(m+1)} \cdot {}_2F_1\left(1, m+1/2; m+1; \frac{1}{1+c}\right) \quad (63)$$

where $c = b/(2a)$ and ${}_2F_1(.,.;.;.)$ denotes the *hypergeometric series* (known also as the *Gauss hypergeometric function*). When m is a positive integer, the integral $J_m(a,b)$ reduces to [3, eq. (7.4.15)], [57, eq. (A13)]

$$J_m(a,b) \triangleq J_m(c) = [P(c)]^m \sum_{k=0}^{m-1} \binom{m-1+k}{k} [1-P(c)]^k \quad (64)$$

where

$$P(x) = \frac{1}{2}\left(1 - \sqrt{\frac{x}{1+x}}\right), \qquad x \geq 0. \quad (65)$$

Using the alternative representation of the Gaussian Q-function (17) in (63), we obtain

$$J_m(a,b) = \frac{a^m}{\Gamma(m)} \int_0^\infty e^{-at} t^{m-1} \left(\frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{bt}{2\sin^2\phi}\right) d\phi\right) dt. \quad (66)$$

Interchanging the order of integration in (66), then using (29), gives

$$J_m(a,b) \triangleq J_m(c) = \frac{1}{\pi}\int_0^{\pi/2} \left(\frac{\sin^2\phi}{\sin^2\phi + c}\right)^m d\phi = \frac{1}{\pi} I_m(c) \quad (67)$$

which is the desired closed-form expression for $I_m(c)$. A similar equivalence can be made between a result derived by Chennakeshu and Anderson [10] and the integrals $\int_0^{(M-1)\pi/M} (\sin^2\phi/\sin^2\phi + c)^m \, d\phi$ and $\int_0^{\pi/M} (\sin^2\phi/\sin^2\phi + c)^m \, d\phi$. Full details on these equivalences can be found in [35, Appendix 5A]. The reason for mentioning these equivalences and the resulting closed-form expressions is that they can be used, for example, to simplify calculations involving the performance BPSK and M-PSK with selection diversity over correlated Nakagami-m fading channels [52].

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