Degrees of Freedom in Adaptive Modulation: A Unified View

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Abstract
We examine adaptive modulation schemes for flat-fading channels where the data rate, transmit power, and instantaneous BER are varied to maximize spectral efficiency, subject to an average power and BER constraint. Both continuous-rate and discrete-rate adaptation are considered, as well as average and instantaneous BER constraints. We find that for a large class of modulation techniques and fading distributions, adapting only the power or only the BER in addition to the data rate yields near-optimal performance. The optimal adaptation of these parameters is a water-filling: increasing the power or decreasing the BER as the channel quality improves. Surprisingly, if the rate is adapted continuously, setting the power and BER to constant values is close to optimal, especially at high SNRs. Hence, the spectral efficiency of adaptive modulation is relatively insensitive to which degrees of freedom are adapted. Therefore, the design of adaptive modulation should be based on practical constraints like implementation complexity and tolerance to fluctuations in BER and transmit power.
I. Introduction

Adaptive modulation is a promising technique to increase spectral efficiency on fading channels. The basic premise of adaptive transmission is a real-time balancing of the link budget through adaptive variation of the transmitted power level, symbol transmission rate, constellation size, coding rate/scheme, or any combination of these parameters [1, 2, 3, 4, 5, 6]. Thus, without wasting power or sacrificing BER, these schemes provide a higher average link spectral efficiency by taking advantage of the time-varying nature of wireless channels: transmitting at high speeds under favorable channel conditions and responding to channel degradation through a smooth reduction of their data throughput. Good performance of adaptive modulation requires accurate channel estimation at the receiver and a reliable feedback path between the receiver and transmitter. Furthermore, since adaptive modulation has a variable transmission rate that depends on the random fluctuations of the channel, adaptive systems may not be well-suited to applications with stringent delay constraints.

Adaptive modulation provides many parameters that can be adjusted relative to the channel fading, including data rate, transmit power, instantaneous BER, symbol rate, and channel code rate/scheme. The question therefore arises as to which of these parameters should be adapted to obtain the best performance. Despite much recent work in adaptive modulation, the performance tradeoffs in adapting each of these degrees of freedom has not been previously examined. Results from [5] indicate that Shannon capacity of a flat-fading channel is achieved by varying the transmission rate and power. This capacity can also be achieved by varying the transmit power alone [9]. However, since capacity is obtained with arbitrarily small BER and the capacity-achieving coding schemes are random, these capacity results are not necessarily applicable to practical adaptive schemes.

In this paper we provide a systematic study on the increase in spectral efficiency obtained by optimally varying combinations of the transmission rate, power, and instantaneous BER. We assume that the resulting adaptive modulation schemes are subject to an average power and BER constraint. We do not consider symbol rate adaptation since it is difficult to implement in real systems. The effect of adaptive channel coding is also not considered. We first analyze adaptive modulation with continuous rate adaptation, where the set of signal constellations is unrestricted, and then we consider the more practical scenario where only a discrete finite set of constellations is available. Analysis is done for both an average and an instantaneous BER constraint. Our goal is to determine the impact on spectral efficiency of adapting various modulation parameters under different constellation restrictions and BER constraints, for a large class of modulation techniques and fading distributions.

The remainder of this paper is organized as follow. The next section describes the system model, including the average power and BER constraints. Section III presents the BER upper bounds used to derive the optimal adaptive modulation scheme. We derive the optimal rate, power, and BER adaptation strategies under different constellation restrictions and BER constraints in Section IV. Plots of average spectral efficiency, optimal power adaptation, and optimal BER adaptation are presented in Section V. Practical considerations in the various adaptive parameters are discussed in Section VI, followed by our conclusions in Section VII.

II. System Model

In this section we present our system model and notation, following that of [6]. The system model is illustrated in Figure 1. We assume a discrete-time channel with stationary and ergodic time-varying gain $\sqrt{g[i]}$ and additive white Gaussian noise $n[i]$. Let $\overline{S}$ denote the average transmit signal power, $N_0$ denote the noise density of $n[i]$, $B$ denote the received signal bandwidth, and $\overline{g}$ denote the average channel power gain. With appropriate scaling of $\overline{S}$, we can assume that $\overline{g} = 1$. There is a feedback path from the receiver to the transmitter for sending channel estimates. This path is assumed be instantaneous and error-free, so the estimation delay $\tau_e = 0$, the feedback delay $\tau_f = 0$, and the channel gain estimate $\hat{g}[i] = g[i]$. The impact of nonzero values for these delay and estimation error parameters has been studied in [6]. For a constant transmit power $\overline{S}$, the instantaneous received SNR
is $\gamma[i] = \mathbb{E}g[i]/(N_0B)$. We denote the transmit power at time $i$ by $S(\gamma[i])$, and the received SNR at time $i$ is then $\gamma[i] (S(\gamma[i]) / S)$. Since $g[i]$ is stationary, the distribution of $\gamma[i]$ is independent of $i$, and we denote this distribution by $p(\gamma)$. When the context is clear, we will omit the time reference $i$ relative to $\gamma$ and $S(\gamma)$. We also assume ideal coherent phase detection.

![System Model](https://example.com/system_model.png)

**Fig. 1. System Model**

The parameters that can be adapted at the transmitter include the transmission rate, power, and instantaneous BER. We will consider both continuous rate adaptation (C-Rate), where the set of signal constellations is unrestricted, as well as discrete rate adaptation (D-Rate), where only a discrete finite set of $N$ constellations is available. For the D-Rate case the rate region boundaries $\{\gamma_i\}_{i=0}^{N}$ define the range of $\gamma$ values over which the different constellations are transmitted. Specifically, we assign one signal constellation and a corresponding data rate of $k_i$ bits/symbol to each rate region $[\gamma_i, \gamma_{i+1})$, where $\gamma_N = \infty$. When the instantaneous SNR $\gamma$ falls within a given region, the associated signal constellation is transmitted. No signal is transmitted if $\gamma \leq \gamma_0$. Thus, $\gamma_0$ serves as a cutoff SNR ratio below which the channel is not used. We will find that in the C-Rate case there is also an optimized cutoff value below which the channel is not used. Thus, for both continuous and discrete rate adaptation, when the channel quality is significantly degraded the channel should not be used.

The spectral efficiency of our modulation scheme equals its data rate per unit bandwidth ($R/B$). When we send $k(\gamma) = \log_2 \lfloor M(\gamma) \rfloor$ bits/symbol, the data rate $R$ is $k(\gamma)/T_s$ bps, where $T_s$ is the symbol time. Assuming Nyquist data pulses ($B = 1/T_s$) we get that for continuous rate adaptation the spectral efficiency is

$$R/B = \int_0^\infty k(\gamma)p(\gamma) d\gamma \text{ bps/Hz} \quad (1)$$

and for discrete rate adaptation it is given by

$$R/B = \sum_{i=0}^{N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma) d\gamma \text{ bps/Hz}. \quad (2)$$

The rate adaptation $k(\gamma)$ is typically parameterized by the average transmit power $\mathbb{E}$ and the BER of the modulation technique, as discussed in more detail in Section IV.

We assume an average transmit power constraint given by

$$\int_0^\infty S(\gamma)p(\gamma) d\gamma \leq \mathbb{E}. \quad (3)$$
For the BER, we assume either an average (A-BER) or an instantaneous (I-BER) constraint. The instantaneous BER constraint implies that every signal constellation must maintain a constant probability of bit error. This is more restrictive than the average constraint. There are two possible definitions for the average BER constraint:

\[
\text{BER} = \frac{E[\text{number of error bits per transmission}]}{E[\text{number of bits per transmission}]} \tag{4}
\]

or

\[
\text{BER} = E \left[ \frac{\text{number of error bits per transmission}}{\text{number of bits per transmission}} \right] \tag{5}
\]

Definition (4) is slightly better than (5) since, for a stationary and ergodic fading process, (4) gives a more accurate measure of the total number of bits received in error divided by the total number of bits received. However, in optimizing the adaptive power and BER, (5) leads to more tractable solutions than (4). We will therefore consider both definitions in deriving the optimal power and BER adaptation.

When applied to continuous rate adaptation (4) becomes

\[
\text{BER} = \frac{\int_{0}^{\infty} \text{BER}(\gamma) k(\gamma) p(\gamma) d\gamma}{\int_{0}^{\infty} k(\gamma) p(\gamma) d\gamma},
\]

whereas (5) yields

\[
\text{BER} = \int_{0}^{\infty} \text{BER}(\gamma) p(\gamma) d\gamma. \tag{7}
\]

When applied to discrete rate adaptation (4) becomes

\[
\text{BER} = \frac{\sum_{i=0}^{N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} \text{BER}(\gamma) p(\gamma) d\gamma}{\sum_{i=0}^{N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma) d\gamma}, \tag{8}
\]

whereas (5) yields

\[
\text{BER} = \sum_{i=0}^{N-1} \int_{\gamma_i}^{\gamma_{i+1}} \text{BER}(\gamma) p(\gamma) d\gamma. \tag{9}
\]

III. BER UPPER BOUNDS

In order to obtain the optimal power and rate adaptation for different modulation schemes, for each modulation technique we need an expression for its BER in AWGN that is easily inverted with respect to rate and power. For most modulation techniques the BER in AWGN is given by the complementary error function \(\text{erfc}\) parameterized by the transmission rate and power. Unfortunately, this function is not easily invertible in its arguments. Therefore, we now introduce tight BER approximations for several modulation techniques that are easily differentiated and inverted. We later use these approximations to derive the optimal power and rate adaptation. For MPSK we consider several different approximations since, as we will see later, although each approximation is tight they lead to very different optimal power adaptations.
The exact expression for the BER of Square MQAM in AWGN as a function of a received SNR \( \gamma S(\gamma)/\bar{S} \) is [7]:

\[
\text{BER}_{MQAM}(\gamma) = \frac{2}{\log_2 M} \left( 1 - \frac{1}{\sqrt{M}} \right) \text{erfc} \left( \sqrt{1.5 \frac{\gamma S(\gamma)}{\bar{S} M - 1}} \right). \tag{10}
\]

We can approximate this expression using the Chernoff upper bound on the erfc function and appropriate constants to get

\[
\text{BER}_{MQAM}(\gamma) \leq 0.2 \exp \left[ \frac{-1.6\gamma (\frac{S(\gamma)}{\bar{S}})}{2^k(\gamma) - 1} \right], \tag{11}
\]

for \( M = 2^k(\gamma) \). This gives an upper bound to BER for \( k(\gamma) \geq 2 \).

For MPAM, the exact BER expression is given as

\[
\text{BER}_{MPAM}(\gamma) = \frac{1}{\log_2 M} \left( 1 - \frac{1}{M} \right) \text{erfc} \left( \sqrt{1.5\gamma \frac{S(\gamma)}{\bar{S} M^2 - 1}} \right). \tag{12}
\]

This expression can also be approximated using the Chernoff upper bound and appropriate constants to get

\[
\text{BER}_{MPAM}(\gamma) \leq 0.1 \exp \left[ -\frac{1.5\gamma \frac{S(\gamma)}{\bar{S}}}{2^{2k(\gamma) - 1}} \right], \tag{13}
\]

which gives an upper bound to BER for \( k(\gamma) \geq 2 \).

For MPSK the exact BER expression is

\[
\text{BER}_{MPSK}(\gamma) = \frac{1}{\log_2 M} \text{erfc} \left( \sqrt{\frac{\gamma S(\gamma)}{\bar{S}}} \sin \left( \frac{\pi}{M} \right) \right). \tag{14}
\]

Here the Chernoff upper bound does not yield a simple and tight upper bound, so we resort to curve-fitting. We find three different BER approximations that are valid for \( k(\gamma) \geq 2 \) are are within 1 dB of error for BER < 10^{-3}. The bounds are given as

Model 1: \( \text{BER}_{MPSK}(\gamma) \approx 0.05 \exp \left[ \frac{-6\gamma \frac{S(\gamma)}{\bar{S}}}{2^{1.9k(\gamma) - 1}} \right] \). \tag{15}

Model 2: \( \text{BER}_{MPSK}(\gamma) \approx 0.2 \exp \left[ \frac{-7\gamma \frac{S(\gamma)}{\bar{S}}}{2^{1.9k(\gamma) + 1}} \right] \). \tag{16}

Model 3: \( \text{BER}_{MPSK}(\gamma) \approx 0.25 \exp \left[ \frac{-8\gamma \frac{S(\gamma)}{\bar{S}}}{2^{1.9k(\gamma)}} \right] \). \tag{17}

These approximations are plotted in Figure 2, where they are shown to be quite tight.
The bounds and approximations for all modulation techniques given above can be written in the following generic form:

$$\text{BER}(\gamma) \approx c_1 \exp \left[ -c_2 \gamma \left( \frac{S(\gamma)}{S} \right) \right],$$

where

$$f(k(\gamma)) = 2^{c_4 k(\gamma)} - c_4,$$

$$c_1, c_2, \text{ and } c_3 \text{ are positive fixed constants, and } c_4 \text{ is a real constant. Note that for the BER bounds on MQAM, MPAM, and MPSK discussed above, } |c_4| \leq 1 \text{ and } 1 \leq c_3 \leq 2. The generic expression (18) is valid for MQAM, MPSK, and MPAM to within 1 dB of error for } k \geq 2 \text{ and } \text{BER} \leq 10^{-3}. \text{ We will see in Section IV that for continuous rate adaptation the optimal power control scheme changes completely depending on the sign of the constant } c_4 \text{ in (19).}

IV. Optimal Rate, Power, and BER Adaptation

In this section we determine the optimal rate, power, and BER adaptation for maximizing spectral efficiency in the following four cases: continuous rate adaptation with an average BER constraint (C-Rate A-BER), continuous rate adaptation with an instantaneous BER constraint (C-Rate I-BER), discrete rate adaptation with an average BER constraint (D-Rate A-BER), and discrete rate adaptation with an instantaneous BER constraint (D-Rate I-BER). Clearly the instantaneous BER constraints are special cases of the average BER constraints, and will therefore have a lower spectral efficiency. Although our derivations are for general fading distributions, we compute our numerical results based on Rayleigh fading. We assume a BER requirement of $10^{-7}$, consistent with typical data requirements (although coded systems would allow much higher BERs). For the discrete rate case we assume 10 different signal constellations are available, corresponding to 2,3,4,5,6,7,8,9,10, and 11 bits/symbol.

A. Continuous Rate and Average BER (C-Rate A-BER)

We now derive the optimal continuous rate, power, and BER adaptation to maximize spectral efficiency (1) subject to the average power constraint (3) and the average BER constraint (6). This
is a standard constrained optimization problem, which we solve using the Lagrange method. The
Lagrange equation is

\[
J(k(\gamma), S(\gamma)) = \int k(\gamma)p(\gamma)d\gamma + \lambda_1 \left[ \int \text{BER}(\gamma)k(\gamma)p(\gamma)d\gamma - \text{BER} \int k(\gamma)p(\gamma)d\gamma \right] + \lambda_2 \left[ \int S(\gamma)p(\gamma)d\gamma - \bar{S} \right].
\] (20)

The optimal rate and power adaptation satisfy

\[
\frac{\partial J}{\partial k(\gamma)} = 0 \quad \text{and} \quad \frac{\partial J}{\partial S(\gamma)} = 0.
\] (21)

Let \(f(k(\gamma))\) be as defined in (19). Then using the generic BER expression (18) in (20) and solving
(21) we get that the power and BER adaptation that maximize spectral efficiency satisfy

\[
\frac{S(\gamma)}{\bar{S}} = \frac{f(k(\gamma))}{\frac{\partial f(k(\gamma))}{\partial k(\gamma)}} \frac{(\lambda_1 \text{BER} - 1)}{\lambda_2 S} - \frac{f(k(\gamma))^2}{c_2 \gamma \left( \frac{\partial f(k(\gamma))}{\partial k(\gamma)} \right) k(\gamma),}
\] (22)

and

\[
\text{BER}(\gamma) = \frac{\lambda_2 S f(k(\gamma))}{\lambda_1 c_2 \gamma k(\gamma)}.
\] (23)

Moreover, from (18), (22), and (23) we get that the optimal rate adaptation \(k(\gamma)\) satisfies

\[
\frac{\text{BER}\lambda_1 - 1}{\frac{\partial f(k(\gamma))}{\partial k(\gamma)}} \lambda_2 S - \frac{f(k(\gamma))}{c_2 \gamma \left( \frac{\partial f(k(\gamma))}{\partial k(\gamma)} \right) k(\gamma)} = \frac{1}{\gamma c_2} \ln \left[ \frac{\lambda_1 c_2 \gamma k(\gamma)}{\lambda_2 S f(k(\gamma))} \right].
\] (24)

Unfortunately, finding a formula for \(k(\gamma)\) that satisfies this equation is quite difficult, since the La-
grangians \(\lambda_1\) and \(\lambda_2\) are based on power and BER averages over the entire fade distribution.

Since the optimal rate adaptation over all \(\gamma\) is hard to determine, let us consider the optimal
adaptation at large values of \(\gamma\). Since our goal is to maximize spectral efficiency, as \(\gamma\) increases we
should be able to send higher data rates, i.e. we expect that

\[
\lim_{\gamma \to \infty} k(\gamma) = \infty.
\] (25)

Under this assumption, since \(|c_4| \leq 1\) and \(1 \leq c_3 \leq 2\) for the modulation schemes we consider, we get that

\[
\lim_{\gamma \to \infty} (2^{c_3 k(\gamma)} - c_4) = 2^{c_3 k(\gamma)}.
\] (26)

Now using (26) in (22) yields

\[
\lim_{\gamma \to \infty} \frac{S(\gamma)}{\bar{S}} = \frac{\lambda_1 \text{BER} - 1}{(\ln 2) S \lambda_2 c_3} - \frac{2^{c_3 k(\gamma)}}{c_2 c_3 (\ln 2) \gamma k(\gamma)}.
\] (27)

Substituting (27) into the RHS of (18) and then setting (23) equal to (18) yields, after some manipu-
lation, that as \(\gamma\) approaches infinity,

\[
\exp \left( \frac{A}{f(k(\gamma))} + \frac{B}{k(\gamma)} \right) = \frac{\tilde{C} f(k(\gamma))}{\gamma k(\gamma)}.
\] (28)
where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are constants that don’t depend on $\gamma$.

Let us now consider the limit of $\frac{f(k(\gamma))}{\gamma k(\gamma)}$ as $\gamma \to \infty$. Since $k(\gamma)$ and $\gamma$ are both monotonically increasing in $\gamma$, this value should converge to a nonnegative constant, which may be zero or infinity. Equation (28) indicates that this limiting value cannot be infinite, since if so then the LHS of (28) goes to zero while the RHS goes to infinity. Therefore

$$\lim_{\gamma \to \infty} \frac{f(k(\gamma))}{\gamma k(\gamma)} = C,$$

where $C$ is a finite nonnegative constant. Substituting (29) into (23) and (27) we see that $S(\gamma)$ and $\text{BER}(\gamma)$ also converge to nonnegative constants as $\gamma$ approaches infinity. This analysis indicates that constant rate and BER are optimal for large $\gamma$ values. However it is not clear how much spectral efficiency is lost by using constant BER and power adaptation at small or moderate $\gamma$ values - this can only be determined by solving for $k(\gamma)$ in (24) (which is quite difficult) and then substituting this into (22). Therefore, we only know the behavior of the optimal power adaptation at large $\gamma$ values. However, we can solve for the optimal power adaptation over all $\gamma$ values under the alternate average BER constraint (7), as we now show.

Under the average BER constraint (7) the Lagrange expression becomes

$$J(k(\gamma), S(\gamma)) = \int k(\gamma)p(\gamma)d\gamma + \lambda_1 \int [\text{BER}(\gamma)p(\gamma)d(\gamma) - \text{BER}] + \lambda_2 \int [S(\gamma)p(\gamma)d(\gamma) - S].$$

The optimal rate and power adaptation satisfy

$$\frac{\partial J}{\partial k(\gamma)} = 0 \text{ and } \frac{\partial J}{\partial S(\gamma)} = 0.$$ 

From (31) and (18) we obtain the following expression for the optimal power adaptation $S(\gamma)$:

$$\frac{S(\gamma)}{S} = -\frac{2^{c_3 k(\gamma)} - c_4}{\lambda_2 c_3 (\ln 2) 2^{c_3 k(\gamma)}},$$

We see from this expression that if $k(\gamma)$ monotonically increases with $\gamma$ then $S(\gamma)$ approaches a constant value at large $\gamma$.

Thus, under both our average BER constraints, we see that the optimal power adaptation becomes constant as $\gamma \to \infty$. This appears to contradict the water-filling power control derived in the next section under an instantaneous BER constraint. But, as we will see in the next section, in Rayleigh fading both water-filling and (32) approach constant values in the limit of large $\gamma$.

The analysis of this section indicates that it is difficult to optimize adaptive rate, power, and BER simultaneously to maximize spectral efficiency. However, we will see in the next section that if BER remains fixed we can easily solve for the optimal power and rate adaptation.

\section*{B. Continuous Rate and Instantaneous BER (C-Rate I-BER)}

We now derive the optimal continuous rate and power adaptation to maximize spectral efficiency (1) subject to the average power constraint (3) and an instantaneous BER constraint $\text{BER}(\gamma) = \text{BER}$. This case was investigated in [6] for MQAM, and we now extend that analysis to more general modulations using our generic BER expression (18). We can invert (18) to express $k(\gamma)$ as a function of the power control $S(\gamma)$ and the fixed bit-error-rate BER as

$$k(\gamma) = \frac{1}{c_3} \log_2 \left[ c_4 - \frac{c_2 \gamma}{\ln (\text{BER}/c_1)} \frac{S(\gamma)}{S} \right].$$
To maximize spectral efficiency (1) we create the Lagrangian

$$J(S(\gamma)) = \int k(\gamma)p(\gamma)d\gamma + \lambda \left[ \int S(\gamma)p(\gamma)d\gamma - S \right].$$

(34)

The optimal power adaptation must satisfy

$$\frac{\partial J}{\partial S(\gamma)} = 0.$$  

(35)

Solving (35) for $S(\gamma)$ yields the optimal power adaptation

$$S(\gamma) = \left\{ \begin{array}{ll}
-\frac{1}{c_3(\ln 2)p_0} + \frac{c_4 \ln \text{BER}/c_1}{c_2} \frac{1}{\gamma} & S(\gamma) \geq 0 \\
0 & \text{else} 
\end{array} \right. ,$$

(36)

which can be written in the more simplified form

$$\frac{S(\gamma)}{S} = \left\{ \begin{array}{ll}
\frac{1}{\gamma_0} - \frac{1}{\gamma K} & S(\gamma) \geq 0 \\
0 & \text{else} 
\end{array} \right. ,$$

(37)

where $K$ is a constant and $\gamma_0$ is an optimized cutoff fade depth below which the channel is not used. This cutoff value must satisfy

$$\int_{\gamma_0}^{\infty} \left( \frac{1}{\gamma_0} - \frac{1}{\gamma K} \right) p(\gamma)d\gamma = 1.$$  

(38)

The value of $K$ depends on the modulation technique, which determines the constants $c_i$, as well as the target BER. For example, MQAM modulation with a target BER of $10^{-7}$ has $K = -1.6/(\ln(5\text{BER})) = 0.11028$. Note that the sign of $K$, which equals the sign of $c_4$ (since $c_1$ and $c_2$ are positive), determines the nature of the power adaptation policy. In particular, if $K$ is positive then (37) uses *waterfilling* in power: more power is used as the channel quality increases above the optimized cutoff fade depth. Conversely, when $K$ is negative the optimal power adaptation is very different, and the transmit power decreases as channel quality increases. When $K$ is infinite ($c_4 = 0$) we get that transmitting at constant power is the best policy. We illustrate the dependence of the optimal power adaptation on $K$ more clearly in Figure 3, where we plot the optimal power adaptation for MPSK using our three different BER models. With Model 1 ($c_4 = 1$) we get a *waterfilling* power control policy, with Model 2 ($c_4 = -1$) we see that at large $\gamma$ values we actually decrease the transmit power as the channel quality increases, and with Model 3 ($c_4 = 0$) we get a constant power policy. Note, however, that as $\gamma$ increases all three policies converge to that of sending constant transmit power, as we also observed in the previous section for the C-Rate A-BER case.

Substituting (37) into (33) yields the optimal rate adaptation

$$k(\gamma) = \frac{1}{c_3} \log_2 \left( \frac{c_4 \gamma K}{\gamma_0} \right).$$

(39)

The corresponding average spectral efficiency is obtained from (1). We plot this average spectral efficiency for MQAM in Rayleigh fading in Figure 4 of Section V. In this figure we also plot the average spectral efficiency obtained using constant transmit power and constant BER. We see that little spectral efficiency is gained using the optimal power adaptation. This can be explained from Figure 3, where we see that the optimal power adaptation converges to constant power as $\gamma$ increases. The optimal power adaptation that achieves the maximum efficiency for MQAM is plotted in Figure 5 in Section V.
C. Discrete Rate and Average BER (D-Rate A-BER)

In the discrete rate case the rate is varied within the set \( k_i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \), and we assign rate \( k_i = i + 2 \) to the rate region \( \gamma_i, \gamma_{i+1} \). Under this rate assignment we wish to maximize spectral efficiency through optimal rate, power, and BER adaptation subject to an average power and BER constraint. Since the set of possible rates and their corresponding rate region assignments are fixed, the optimal rate adaptation corresponds to finding the optimal rate region boundaries \( \gamma_i, i = 0, \ldots, N-1 \). The Lagrangian for this constrained optimization problem is

\[
J(\gamma_1, \gamma_2, \ldots, \gamma_N, S(\gamma)) = \sum_{0 \leq i \leq N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma) d\gamma + \lambda_1 \left[ \sum_{0 \leq i \leq N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} (BER(\gamma) - \overline{BER}) p(\gamma) d\gamma \right] + \lambda_2 \left[ \int S(\gamma) p(\gamma) d\gamma - S \right]
\]

(40)

The optimal power adaptation is obtained by solving the following equation for \( S(\gamma) \):

\[
\frac{\partial J}{\partial S(\gamma)} = 0.
\]

(41)

Similarly, the optimal rate region boundaries are obtained by solving the following set of equations for \( \gamma_i \).

\[
\frac{\partial J}{\partial \gamma_i} = 0, \quad 0 \leq i \leq N - 1.
\]

(42)

From (41) we see that the optimal power and BER adaptation must satisfy

\[
\frac{\partial BER(\gamma)}{\partial S(\gamma)} = -\frac{\lambda_2}{k_i \lambda_1}, \quad \gamma_i \leq \gamma \leq \gamma_{i+1}.
\]

(43)
Substituting (18) into (43) we get that

$$\text{BER}(\gamma) = \lambda \frac{f(k_i)}{\gamma k_i},$$

(44)

where $\lambda = \frac{3A_2}{c_2 \lambda_1}$. This form of BER adaptation can also thought of as BER 
*waterfilling* since the BER decreases when the channel quality improves.

Now setting the BER in (18) equal to (44) and solving for $S(\gamma)$ yields

$$S(\gamma) = S_i(\gamma), \quad \gamma_i \leq \gamma \leq \gamma_{i+1}$$

(45)

where

$$\frac{S_i(\gamma)}{S} = \ln \left[ \frac{\lambda f(k_i)}{c_1 \gamma k_i} \right] \frac{f(k_i)}{e^{-\gamma c_2}}, \quad 0 \leq i \leq N - 1,$$

(46)

and $S(\gamma) = 0$ for $\gamma < \gamma_0$. We see from (46) that $S(\gamma)$ is discontinuous at the $\gamma_i$ boundaries.

Now let’s look into the optimal rate region boundaries. From (42) we get that

$$\text{BER}(\gamma_i) = \text{BER} - \frac{1}{\lambda_1} - \frac{\lambda_2}{\lambda_1} \frac{S_i(\gamma_i) - S_{i-1}(\gamma_i)}{k_i - k_{i-1}}, \quad 0 \leq i \leq N - 1.$$  

(47)

Unfortunately, this set of equations is very difficult to solve for the optimal $\gamma_i$. However, if we assume that $S(\gamma)$ is continuous at each boundary then we get that

$$\text{BER}(\gamma_i) = \text{BER} - \frac{1}{\lambda}, \quad 0 \leq i \leq N - 1.$$  

(48)

for some constant $\lambda$. Under this assumption we can solve for the suboptimal rate region boundaries as

$$\gamma_i = \frac{f(k_i)}{k_i}, \quad 0 \leq i \leq N - 1,$$

(49)

for some constant $\rho$. The constants $\lambda$ and $\rho$ are found numerically such that the average power and BER constraints are satisfied. The average spectral efficiency for this D-Rate A-BER case using the suboptimal rate region boundaries (49) is plotted in Figure 4 in Section V. We see that this efficiency is very close to the efficiency when rate is adapted continuously (we expect that if we used a smaller set of discrete rates we would see a more significant efficiency penalty.) The power adaptation $S(\gamma_i)$ for the rate region boundaries of (49) is plotted in Figure 5 of Section V. We see from this figure that in general power increases with $\gamma$, however within each rate region we transmit the maximum power at the lower region boundary $\gamma_i$, and then the transmit power decreases for $\gamma_i < \gamma < \gamma_{i+1}$. The BER adaptation (44) for the rate region boundaries of (49) is shown in Figure 6 of Section V. We see from this figure that the fluctuations of BER about its mean value are insignificant, which indicates that the spectral efficiency for the D-Rate I-BER case should be about the same as the D-Rate A-BER case. This will be demonstrated in the next section.

**D. Discrete Rate and Instantaneous BER (D-Rate I-BER)**

We now assume the same discrete rate adaptation as in the previous section and an instantaneous BER constraint, so that $\text{BER}(\gamma) = \text{BER}$. Under these constraints the optimal power adaptation is given as

$$\frac{S(\gamma)}{S} = \frac{h(k_i)}{\gamma}$$

(50)
where \( h(k_i) = -\frac{1}{c_i} \ln \left( \frac{\text{BER}}{\gamma_i} \right) f(k_i) \).

We find the optimal rate region boundaries that maximize spectral efficiency using the Lagrangian method. The Lagrange equation is given as

\[
J(\gamma_1, \gamma_2, ..., \gamma_N) = \sum_{0 \leq i \leq N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma) d\gamma + \lambda \sum_{0 \leq i \leq N-1} \int_{\gamma_i}^{\gamma_{i+1}} \frac{h(k_i)}{\gamma} p(\gamma) d\gamma. \tag{51}
\]

The optimal rate region boundaries are obtained by solving the following equation for \( \gamma_i \).

\[
\frac{\partial J}{\partial \gamma_i} = 0, \quad 0 \leq i \leq N - 1. \tag{52}
\]

This yields

\[
\gamma_0 = \frac{h(k_1)}{k_1} \rho \tag{53}
\]

and

\[
\gamma_i = \frac{h(k_i) - h(k_{i-1})}{k_i - k_{i-1}} \rho, \quad 1 \leq i \leq N - 1 \tag{54}
\]

where \( \rho \) is determined by the average power constraint. The spectral efficiency and optimal power adaptation for this case are plotted, respectively, in Figures 4 and 5 in Section V. We see that the spectral efficiency in this case is close to that in the other three cases. To maintain a constant BER in each rate region we use less power when channel is good.

V. Numerical Results

In this section we plot the average spectral efficiency, optimal power adaptation, and optimal BER adaptation for the C-Rate I-BER, D-Rate A-BER, and D-Rate I-BER cases, as well as for the C-Rate case with constant transmit power and BER. The average spectral efficiency for these cases is shown in Figure 4. The optimal power adaptation is shown in Figure 5. The optimal BER adaptation is shown in Figure 6. Surprisingly, the difference in spectral efficiency between the four cases is negligible, indicating that restricting the rate adaptation to discrete sets results in little performance loss, and that adapting the instantaneous power and BER yields little performance gain.

VI. Practical Considerations

This section discusses practical considerations in the design of adaptive modulation schemes. Since our results in Section V indicate that C-Rate A-BER, C-Rate I-BER, D-Rate A-BER and D-Rate I-BER all have approximately the same spectral efficiency, other performance measures and implementation complexity should drive the adaptive modulation design. Under an average BER constraint the BER fluctuates about its mean value. If the fluctuations are significant then this can lead to large error bursts. These error bursts can be controlled using an interleaver with error correction coding, but this typically adds complexity and delay to the system design. Power and rate variation at the transmitter impose additional complexity on the transmitter hardware. Finally, continuous rate adaptation is not feasible in practice. We can approximate the C-Rate case using a large number of discrete constellations, but the transmitter and receiver hardware complexity grows as the number of signal constellations increases. We did not investigate the spectral efficiency in the D-Rate case for a small number of signal constellations. For the discrete set we did consider, a greater variation of BER and power was needed to maximum throughput than in the C-Rate case. Also, in the D-Rate case we find that outage occurs at a lower SNR value with the instantaneous BER constraint (13dB) than with the average BER constraint (18dB).
VII. Conclusion

We have shown that the maximum spectral efficiency of adaptive modulation schemes is nearly the same under continuous and discrete rate adaptation as well as an instantaneous or average BER constraint. We have also derived the optimal power adaptation for these schemes for a large class of modulation techniques and general fading distributions. We find that setting the power and BER constant achieves near optimal performance, and therefore practical adaptive modulation schemes should focus on optimizing their rate adaptation.

References

Fig. 6. $BER(\gamma)$ for MQAM @BER = $10^{-7}$, $\gamma = 30dB$


